ON THE MODULI OF CONSTANT MEAN CURVATURE CYLINDERS OF FINITE TYPE IN THE 3-SPHERE

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Abstract. We show that one-sided Alexandrov embedded constant mean curvature cylinders of finite type in the 3-sphere are surfaces of revolution. This confirms a conjecture by Pinkall and Sterling that the only embedded constant mean curvature tori in the 3-sphere are rotational.

Introduction

Alexandrov [4] proved that there are no compact embedded surfaces with constant mean curvature (cmc) in Euclidean 3-space $\mathbb{R}^3$ other than round spheres. However, while there are no compact minimal surfaces in $\mathbb{R}^3$, there is an abundance of such in the 3-sphere $\mathbb{S}^3$. For instance 2-spheres in the 3-sphere are minimal precisely when they are great 2-spheres, and Lawson proved that compact embedded minimal surfaces in $\mathbb{S}^3$ exist for every genus [42, 43]. Lawson further showed [44] that any embedded minimal torus in $\mathbb{S}^3$ is unknotted, and conjectured that up to isometry the Clifford torus is the only embedded minimal torus in $\mathbb{S}^3$. Hsiang and Lawson [27] proved that the only embedded minimal torus of revolution is the Clifford torus. Further results suggest that an embedded minimal torus indeed has additional symmetries: Montiel and Ros [49] showed that the only minimal torus immersed into $\mathbb{S}^3$ by the first eigenfunctions is the Clifford torus, and Ros [56] proved that the normal surface of an embedded minimal torus in $\mathbb{S}^3$ is also embedded. Various methods for obtaining minimal surfaces in $\mathbb{S}^3$ have been employed to study specific classes, as in Karcher, Pinkall and Sterling [34], and more recently by Kapouleas and Yang [33].

Wente's discovery [71] of cmc tori provided the first compact examples other than spheres in Euclidean 3-space. The studies of Abresch [1, 2], Wente [72] and Walter [70] on special classes of cmc tori in $\mathbb{R}^3$ concluded in the classification by Pinkall and Sterling [51], and their algebro-geometric description by Bobenko [5, 6]. In fact, Bobenko gave explicit formulas for cmc tori in $\mathbb{R}^3$, $\mathbb{S}^3$ and hyperbolic 3-space $\mathbb{H}^3$ in terms of theta–functions, and provided a unified description of cmc tori in the 3-spaceforms in terms of algebraic curves and spectral data. Independently, Hitchin [25] classified harmonic 2-tori in the 3-sphere, and thus also as a special case the harmonic Gauss maps of cmc tori. These ideas culminated in the description of harmonic tori in symmetric spaces by Burstall, Ferus, Pedit and Pinkall [9], and the generalized Weierstraß representation by Dorfmeister, Pedit and Wu [17].

Associated to a cmc torus in the 3-sphere is a hyperelliptic Riemann surface, the so called spectral curve. The structure equation for cmc tori is the sinh-Gordon equation. Hitchin [25], and Pinkall and Sterling [51] independently proved that all doubly periodic solutions of the sinh-Gordon equation correspond to spectral curves of finite genus. The genus of the spectral curve is called spectral genus. Ercolani, Knörrer and Trubowitz [18] proved that for every even spectral genus $g \geq 2$ there exists a hyperelliptic curve which corresponds to an immersed cmc torus in $\mathbb{R}^3$. The remaining cases of odd genera $g > 1$ was settled by Jaggy [29]. Adapting these results, Carberry [13] showed that minimal tori in $\mathbb{S}^3$ exist for every spectral genus. Note

*Mathematics Subject Classification.*53A10, 53C17. May 17, 2008.

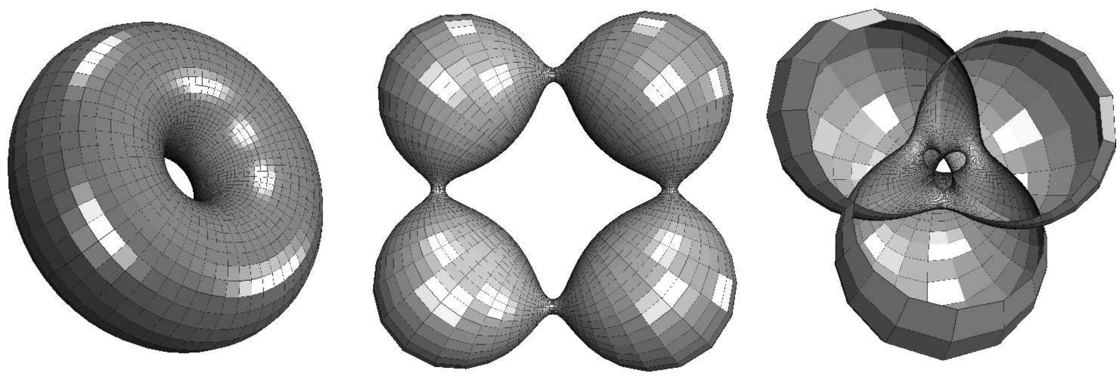

FIGURE 1. Stereographic projections of constant mean curvature tori of revolution in $\mathbb{S}^3$: On the left the Clifford torus, an embedded minimal torus with spectral genus 0. In the middle, an embedded non-minimal torus of spectral genus 1. On the right, a cutaway view of a non-embedded minimal torus of spectral genus 1. Images were created with cmclab [59].

that while a CMC torus in $\mathbb{R}^3$ has at least spectral genus 2, there is no such restriction for CMC tori in $\mathbb{S}^3$. In particular, CMC tori of revolution in $\mathbb{S}^3$ have spectral genus $g \leq 1$. Pinkall and Sterling [51] conjectured that the only embedded CMC tori in $\mathbb{S}^3$ are tori of revolution.

In parallel the global theory of embedded CMC (especially minimal) surfaces in space forms was developed using geometric PDE methods. Meeks [47] proved that a properly embedded end of a CMC surface is cylindrically bounded, which was used by Korevaar, Kusner and Solomon [39] to prove that the only embedded CMC cylinders in $\mathbb{R}^3$ are surfaces of revolution - either a standard round cylinder (spectral genus $g = 0$), or a Delaunay unduloid (spectral genus $g = 1$). There are analogous CMC surfaces of revolution in $\mathbb{S}^3$, some of which close up into tori, see Figure 1 for some simple examples.

Kapouleas [30, 31, 32] proved the existence of compact CMC surfaces in $\mathbb{R}^3$ for any genus greater than 1, as well as many new classes of non-compact CMC surfaces in $\mathbb{R}^3$, but not much is known about the moduli of CMC surfaces in general. Progress on understanding the moduli of CMC immersions of punctured spheres has recently been made in the case of three punctures by Grosse-Brauckmann, Kusner and Sullivan [23] and by Schmitt et. al. [60]. Kusner, Mazzeo and Pollack [41] show that the moduli space of CMC surfaces is an analytic variety. The local linearization of the moduli space is described by Jacobi fields which correspond to a normal variation of the surface which preserve the constant mean curvature property. Recently Korevaar, Kusner and Ratzkin [38] studied Jacobi fields on a class of CMC surfaces with the additional property of being Alexandrov immersed. An Alexandrov immersed surface in $\mathbb{R}^3$ is a complete noncompact properly immersed surface $f : \Sigma \to \mathbb{R}^3$ that is the boundary of a 3-manifold M with two additional features: The mean curvature normal of Σ points into M, and f extends to a proper immersion of M into $\mathbb{R}^3$. When the target is the 3-sphere, we replace properness by completeness, and as Lawson [44] we consider in analogy a smooth immersion $f : \Sigma \to \mathbb{S}^3$ that we call a one-sided Alexandrov embedding if Σ is the boundary of a connected 3-manifold M and the following two conditions hold: The mean curvature of Σ with respect to the inward normal is non-negative. Secondly, the manifold M is complete with respect to the metric induced by f. We prove that the property of one-sided Alexandrov embeddedness is stable under continuous deformation, which allows us to study continuous families of one-sided Alexandrov embedded surfaces.

In this paper we consider CMC cylinders which have constant Hopf differential, and whose metric is a periodic solution of the sinh-Gordon equation of finite type. Such CMC cylinders are said to be of finite type. We describe such finite type CMC cylinders in Section 3 and 4 by spectral data, and show that the spectral data can be deformed in such a way that the corresponding family of CMC surfaces are all topologically cylinders. It turns out that the corresponding moduli space of spectral data of genus g is $g+1$-dimensional. Furthermore, we can control the spectral genus under the deformation, and by successively coalescing branch points of the spectral curve, we continuously deform the spectral curve in Lemma 4.5 into a curve of genus zero. In Section 5 we show that one-sided Alexandrov embedded surfaces with constant mean curvature have collars with depths uniformly bounded from below. For this purpose we use a 'maximum principle at infinity' which was communicated to us by Harold Rosenberg [57]. This allows us to show in Theorem 6.8 that a large class of continuous deformations of CMC cylinders of finite type preserve the one-sided Alexandrov embeddedness. In Lemma 7.3 we continuously deform any one-sided finite type CMC cylinder in $\mathbb{S}^3$ into a one-sided Alexandrov embedded flat cylinder in $\mathbb{S}^3$ with spectral genus zero. These are classified in Theorem 7.1. Finally this classification is extended to all possible deformations of these flat CMC cylinders in $\mathbb{S}^3$ in Theorem 7.6. Since an embedded CMC torus in the 3-sphere is covered by a one-sided Alexandrov embedded cylinder, our result confirms the conjecture by Pinkall and Sterling, and since the only embedded minimal torus of revolution is the Clifford torus, also affirms Lawson's conjecture.

Acknowledgments. We thank Fran Burstall, Karsten Grosse-Brauckmann, Ian McIntosh, Rob Kusner, Franz Pedit and Ulrich Pinkall for useful discussions. This work was mostly carried out whilst M Kilian was a research assistant at the University of Mannheim, and he would like to thank the Institute of Mathematics there for providing excellent research conditions.

We had several beneficial conversations with Laurent Hauswirth, and we are especially grateful to Antonio Ros and Harold Rosenberg who helped us close a gap in a first draft of this paper, and Univerité Paris 7 for its hospitality during these discussions.

1. Conformal CMC immersions into $\mathbb{S}^3$

This preliminary section recalls the relationship between CMC immersed surfaces in $\mathbb{S}^3$ and solutions of the sinh-Gordon equation, before considering the special case of CMC cylinders in $\mathbb{S}^3$, the notion of monodromy and the period problem.

1.1. The sinh-Gordon equation.

We identify the 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ with $\mathbb{S}^3 \cong \mathrm{SU}_2$. The Lie algebra of the matrix Lie group SU_2 is $\mathfrak{su}_2$, equipped with the commutator $[\cdot, \cdot]$. For $\alpha,\, \beta \in \Omega^1(\mathbb{R}^2, \mathfrak{su}_2)$ smooth 1–forms on $\mathbb{R}^2$ with values in $\mathfrak{su}_2$, we define the $\mathfrak{su}_2$–valued 2–form

$$[\alpha \wedge \beta](X,\, Y) = [\alpha(X),\, \beta(Y)] - [\alpha(Y),\, \beta(X)],$$

for vector fields X, Y on $\mathbb{R}^2$. Let $L_g : h \mapsto gh$ be left multiplication in SU_2. Then by left translation, the tangent bundle is $T\mathrm{SU}_2 \cong \mathrm{SU}_2 \times \mathfrak{su}_2$ and $\theta : T\mathrm{SU}_2 \to \mathfrak{su}_2,\, v_g \mapsto dL_{g^{-1}}(v_g)$ is the (left) Maurer–Cartan form. It satisfies the **Maurer-Cartan-equations**

$$2\, d\theta + [\theta \wedge \theta] = 0. \tag{1.1}$$

For a map $F : \mathbb{R}^2 \to \mathrm{SU}_2$, the pullback $\alpha = F^*\theta$ also satisfies (1.1), and conversely, every solution $\alpha \in \Omega^1(\mathbb{R}^2, \mathfrak{su}_2)$ of (1.1) integrates to a smooth map $F : \mathbb{R}^2 \to \mathrm{SU}_2$ with $\alpha = F^*\theta$.

Complexifying the tangent bundle $T\mathbb{R}^2 \cong T\mathbb{C}$ and decomposing into $(1,0)$ and $(0,1)$ tangent spaces, and writing $d = \partial + \bar\partial$, we may split $\omega \in \Omega^1(M, \mathfrak{su}_2)$ into the $(1,0)$ part ω', the $(0,1)$ part ω'' and write $\omega = \omega' + \omega''$. We set the $*$–operator on $\Omega^1(M, \mathfrak{sl}_2)$ to $*\omega = -i\omega' + i\omega''$.

We denote by $\langle \cdot\,,\cdot \rangle$ the bilinear extension of the Ad–invariant inner product $(X,\,Y) \mapsto -\frac{1}{2}\mathrm{tr}(XY)$ of $\mathfrak{su}_2$ to $\mathfrak{su}_2^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$. The double cover of the isometry group $\mathrm{SO}(4)$ is $\mathrm{SU}_2 \times \mathrm{SU}_2$ via the action $((F,G),X) \mapsto FXG^{-1}$.

Now let $g : \mathbb{R}^2 \to \mathrm{SU}_2$ be an immersion and $\omega = g^{-1}dg = \omega' + \omega''$. Then g is conformal if and only if the $(1,0)$-part of ω is isotropic

$$\langle \omega',\, \omega' \rangle = 0\,. \tag{1.2}$$

If g is a conformal immersion then there exists a smooth function $v : \mathbb{R}^2 \to \mathbb{R}^\times = \mathbb{R}\setminus\{0\}$, called the **conformal factor** of g such that

$$v^2 = 2\,\langle \omega',\, \omega'' \rangle\,. \tag{1.3}$$

The mean curvature function H of g (see e.g. [60]) is given by

$$2\,d * \omega = H\,[\omega \wedge \omega]. \tag{1.4}$$

Recall the following observation of Uhlenbeck [68], based on an earlier result by Pohlmeyer [52], and suppose that in the following $g : \mathbb{R}^2 \to \mathrm{SU}_2$ is a conformal immersion with non-zero constant mean curvature H_0 and conformal factor v. Then (1.4) and $2d\omega + [\,\omega\wedge\omega\,] = 0$ combined give $d\omega + H_0^{-1} d * \omega = 0$, or alternatively

$$(1 - iH_0^{-1})\,d\omega' + (1 + iH_0^{-1})\,d\omega'' = 0. \tag{1.5}$$

Inserting $d\omega'' = -d\omega' - [\,\omega' \wedge \omega''\,]$ respectively $d\omega' = -d\omega'' - [\,\omega' \wedge \omega''\,]$ into (1.5) gives $2d\omega' = (iH_0 - 1)[\,\omega' \wedge \omega''\,]$ and $2d\omega'' = -(1 + iH_0)[\,\omega' \wedge \omega''\,]$. Then an easy computation shows that

$$\alpha_\lambda = \tfrac{1}{2}(1 + \lambda^{-1})(1 + iH_0)\,\omega' + \tfrac{1}{2}(1 + \lambda)(1 - iH_0)\,\omega''$$

satisfies the Maurer-Cartan-equations

$$2\,d\alpha_\lambda + [\,\alpha_\lambda \wedge \alpha_\lambda\,] = 0 \text{ for all } \lambda \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

The Maurer-Cartan-equations are an integrability condition, so we can integrate and obtain a corresponding **extended frame** $F_\lambda : \mathbb{R}^2 \times \mathbb{C}^\times \to \mathrm{SL}_2(\mathbb{C})$ with $dF_\lambda = F_\lambda\,\alpha_\lambda$ and $F_\lambda(0) = \mathbb{1}$. Since ω takes values in $\mathfrak{su}_2$, we conclude that F_λ takes values in SU_2 when $\lambda \in \mathbb{S}^1$. Now define for $\lambda_0,\,\lambda_1 \in \mathbb{S}^1$, $\lambda_0 \neq \lambda_1$, the following map $f : \mathbb{R}^2 \to \mathrm{SU}_2$ by the **Sym-Bobenko-formula**

$$f = F_{\lambda_1} F_{\lambda_0}^{-1}\,. \tag{1.6}$$

Then for $\Omega = f^{-1}df = \mathrm{Ad}\,F_{\lambda_0}\,(\alpha_{\lambda_1} - \alpha_{\lambda_0})$ we obtain $\Omega' = \frac{1}{2}(\lambda_1^{-1} - \lambda_0^{-1})(1 + iH_0)\,\mathrm{Ad}\,F_{\lambda_0}\,\omega'$ so the conformality of f follows from the conformality of g, since

$$\langle \Omega',\, \Omega' \rangle = \tfrac{1}{4}(\lambda_1^{-1} - \lambda_0^{-1})^2(1 + iH_0)^2\,\mathrm{Ad}\,F_{\lambda_0}\,\langle \omega',\, \omega' \rangle = 0 \tag{1.7}$$

$$2\,\langle \Omega',\, \Omega'' \rangle = \sin^2(t_1 - t_0)(1 + H_0^2)\,v^2\,. \tag{1.8}$$

Here we have written $\lambda_{0,1} = e^{2it_{0,1}}$, and v is the conformal factor of g. Furthermore

$$d * \Omega = \tfrac{i}{4}(1 + H_0^2)\left(\lambda_0^{-1}\lambda_1 - \lambda_0\lambda_1^{-1}\right)\mathrm{Ad}\,F_{\lambda_0}[\omega' \wedge \omega'']\,,$$
$$[\Omega \wedge \Omega] = \tfrac{1}{2}(1 + H_0^2)(\lambda_1^{-1} - \lambda_0^{-1})(\lambda_1 - \lambda_0)\,\mathrm{Ad}\,F_{\lambda_0}\,[\omega' \wedge \omega'']\,.$$

Hence by (1.4), the map $f : \mathbb{R}^2 \to \mathrm{SU}_2$ given by (1.6) has constant mean curvature

$$H = i\,\frac{\lambda_0 + \lambda_1}{\lambda_0 - \lambda_1}\,. \tag{1.9}$$

In summary, by starting with one non-minimal conformal CMC immersion g, we have just seen how to obtain a whole $\mathbb{C}^\times$-family of solutions of the Maurer-Cartan-equations, and from the corresponding extended frame we then obtained another conformal CMC immersion f. Since the mean curvature (1.9) and the conformal factor of f in (1.8) only depend on the angle between $\lambda_0,\,\lambda_1$, we in fact get a whole $\mathbb{S}^1$-family of isometric conformal CMC immersions, called

an **associated family**, which is obtained by simultaneously rotating λ_0, λ_1 while keeping the angle between them fixed.

We next recall the following version of Theorem 14.1 in Bobenko [6], which provides a correspondence between solutions of the sinh-Gordon equation and associated families of CMC surfaces in the 3-sphere.

Theorem 1.1. [6] *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and define*

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z\,dz - u_{\bar z}\,d\bar z & i\,\lambda^{-1}e^u\,dz + i\,e^{-u}\,d\bar z \\ i\,e^{-u}\,dz + i\,\lambda\,e^u\,d\bar z & -u_z\,dz + u_{\bar z}\,d\bar z \end{pmatrix}. \tag{1.10}$$

Then $2\,d\alpha_\lambda + [\,\alpha_\lambda \wedge \alpha_\lambda\,] = 0$ if and only if u is a solution of the sinh-*Gordon equation*

$$\partial\bar\partial\, 2u + \sinh(2u) = 0. \tag{1.11}$$

For any solution u of the sinh-*Gordon equation and corresponding extended frame F_λ, and λ_0, $\lambda_1 \in \mathbb{S}^1$, $\lambda_0 \neq \lambda_1$, the map defined by the Sym-Bobenko-formula* (1.6) *is a conformal immersion with constant mean curvature H* (1.9)*, conformal factor $v = e^u/\sqrt{H^2+1}$, and constant Hopf differential $Q\,dz^2$ with $Q = i\,(\lambda_1^{-1} - \lambda_0^{-1})/4$.*

Proof. Decomposing $\alpha_\lambda = \alpha'_\lambda\,dz + \alpha''_\lambda\,d\bar z$ into (1, 0) and (0, 1) parts, we compute

$$\bar\partial\alpha'_\lambda = \frac{1}{2}\begin{pmatrix} u_{z\bar z} & i\lambda^{-1}u_{\bar z}e^u \\ -iu_{\bar z}e^{-u} & -u_{z\bar z}\end{pmatrix}, \quad \partial\alpha''_\lambda = \frac{1}{2}\begin{pmatrix} -u_{z\bar z} & -iu_z e^{-u} \\ i\lambda u_z e^u & u_{z\bar z}\end{pmatrix},$$

$$[\alpha'_\lambda,\,\alpha''_\lambda] = \frac{1}{4}\begin{pmatrix} -e^{2u} + e^{-2u} & 2iu_{\bar z}\lambda^{-1}e^u + 2iu_z e^{-u} \\ -2i\lambda u_z e^u - 2iu_{\bar z}e^{-u} & e^{2u} - e^{-2u}\end{pmatrix}.$$

Now $2\,d\alpha_\lambda + [\,\alpha_\lambda \wedge \alpha_\lambda\,] = 0$ is equivalent to $\bar\partial\alpha'_\lambda - \partial\alpha''_\lambda = [\alpha'_\lambda,\,\alpha''_\lambda]$, which holds if and only if u solves the sinh-Gordon equation (1.11).

If u is a solution of the sinh-Gordon equation, then we may integrate $dF_\lambda = F_\lambda\,\alpha_\lambda$ to obtain a map $F_\lambda : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathrm{SU}_2$. Let λ_0, $\lambda_1 \in \mathbb{S}^1$, $\lambda_0 \neq \lambda_1$, and consider the map $f : \mathbb{R}^2 \to \mathrm{SU}_2$ defined by the Sym-Bobenko-formula (1.6). Conformality (1.2) is a consequence of the fact that the complexified tangent vector

$$f^{-1}\partial f = \mathrm{Ad}\,F_{\lambda_0}(\alpha'_{\lambda_1} - \alpha'_{\lambda_0}) = \frac{i}{2}\,e^u\,(\lambda_1^{-1} - \lambda_0^{-1})\,\mathrm{Ad}\,F_{\lambda_0}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0\end{smallmatrix}\right)$$

is isotropic with respect to the bilinear extension of the Killing form. The mean curvature can be computed using formula (1.4). The conformal factor is obtained from

$$v^2 = 2\,\langle f^{-1}\partial f,\, f^{-1}\bar\partial f\rangle = \tfrac{1}{4}\,e^{2u}\,(\lambda_1^{-1} - \lambda_0^{-1})(\lambda_1 - \lambda_0).$$

From (1.9) we have $(H^2+1)(\lambda_1^{-1} - \lambda_0^{-1})(\lambda_1 - \lambda_0) = 4$, which proves the formula for the conformal factor.

Define the normal $N = F_{\lambda_1}\,\varepsilon\,F_{\lambda_0}^{-1}$ with $\varepsilon = \left(\begin{smallmatrix} i & 0 \\ 0 & -i\end{smallmatrix}\right)$. Then $\partial N = F_{\lambda_1}(\alpha'_{\lambda_1}\,\varepsilon - \varepsilon\,\alpha'_{\lambda_0})\,F_{\lambda_0}^{-1}$ and

$$\alpha'_{\lambda_1}\,\varepsilon - \varepsilon\,\alpha'_{\lambda_0} = \begin{pmatrix} 0 & \frac{1}{2}e^u(\lambda_1^{-1} + \lambda_0^{-1}) \\ -e^{-u} & 0\end{pmatrix}.$$

Consequently, $Q := -\langle\partial\partial f,\, N\rangle = \langle\partial f,\, \partial N\rangle = \langle F_{\lambda_1}^{-1}\,\partial f\,F_{\lambda_0},\, F_{\lambda_1}^{-1}\,\partial N\,F_{\lambda_0}\rangle = \frac{i}{4}\,(\lambda_1^{-1} - \lambda_0^{-1})$, which proves the formula for the Hopf differential, and concludes the proof. $\square$

There is an analogous but more general theorem (see e.g Bobenko [7]) than the one above, which asserts that if functions (u, Q, $H \equiv$ *const.*) satisfy the Gauss-Codazzi equations, then one obtains a $\mathbb{C}^\times$-family of solutions of the Maurer-Cartan-equations, thus an extended frame and consequently an associated family via the Sym-Bobenko-formula.

1.2. **Monodromy and periodicity condition.** The CMC condition implies that the Hopf differential is a holomorphic quadratic differential [26]. On the cylinder $\mathbb{C}^\times$ there is an infinite dimensional space of holomorphic quadratic differentials, large classes of which can be realized as Hopf differentials of CMC cylinders [36]. On a CMC torus the Hopf differential is constant (and non-zero). Since we are ultimately interested in tori, we restrict our attention to CMC cylinders considered via Theorem 1.1 which have constant non-zero Hopf differentials on the universal covering $\mathbb{C}$ of $\mathbb{C}^\times$. Note that for given solution u of the sinh-Gordon equation an extended frame $\lambda \mapsto F_\lambda$ is holomorphic on $\mathbb{C}^\times$ and has essential singularities at $\lambda = 0, \infty$.

Let F_λ be an extended frame for a CMC immersion $f : \mathbb{R}^2 \to \mathbb{S}^3$ such that (1.6) holds for two distinct unimodular numbers λ_0, λ_1. Let $\tau : \mathbb{R}^2 \to \mathbb{R}^2$, $z \mapsto z + \tau$ be a translation, and assume that $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$ has period τ, so that $\tau^* \alpha_\lambda = \alpha_\lambda \circ \tau = \alpha_\lambda$. Then we define the **monodromy** of F_λ with respect to τ as

$$M_\lambda(\tau) = \tau^*(F_\lambda)\, F_\lambda^{-1} \,. \tag{1.12}$$

Periodicity $\tau^* f = f$ in terms of the monodromy is then $\tau^* f = M_{\lambda_1}(\tau) F_{\lambda_1} F_{\lambda_0}^{-1} M_{\lambda_0}^{-1}(\tau)$, so $\tau^* f = f$ if and only if

$$M_{\lambda_0}(\tau) = M_{\lambda_1}(\tau) = \pm \mathbb{1} \,. \tag{1.13}$$

If $\Delta(\lambda)$ is the trace of $M_\lambda(\tau)$ then $\tau^* f = f$ if and only if $\Delta(\lambda_0) = \Delta(\lambda_1) = \pm 2$.

2. Finite type solutions of the sinh-Gordon equation

In this section we introduce the solutions of the sinh-Gordon equation which are called finite type solutions. Finite type solutions of the sinh-Gordon equation are in one-to-one correspondence with maps called polynomial Killing fields. These polynomial Killing fields take values in certain 2×2-matrix polynomials, and solve a non-linear partial differential equation, but they are uniquely determined by one of their values. We shall call these values initial values of polynomial Killing fields or just initial values. The Symes method calculates the solutions in terms of the initial values with the help of a loop group splitting. The eigenvalues of these matrix polynomials define a real hyperelliptic algebraic curve, which is called **spectral curve**. One spectral curve corresponds to a whole family of finite type solutions of the sinh-Gordon equation. We call the sets of finite type solutions (or their initial values), which belong to the same spectral curve, isospectral sets. The eigenspaces of the matrix polynomials define a holomorphic line bundle on the spectral curves called eigenbundle. These holomorphic line bundles completely determine the corresponding initial value and the corresponding solution of the sinh-Gordon equation. Consequently, the isospectral sets can be identified with one connected component of the real part of the Picard group. In case the spectral curve has singularities, then the isospectral set can be identified with the real part of the compactification of a generalized Jacobian. These compactifications have stratifications, whose strata are the orbits under the action of the generalized Jacobian. In our case the spectral curves are hyperelliptic and we shall describe the corresponding stratifications of the isospectral sets.

2.1. **Polynomial Killing fields.** For some aspects of the theory untwisted loops are advantageous, and avoiding the additional covering map $\lambda \mapsto \sqrt{\lambda}$ simplifies for example the description of Bianchi-Bäcklund transformations by the simple factors [66, 37]. For the description of polynomial Killing fields on the other hand, the twisted loop algebras as in [9, 11, 12, 17, 46] are better suited, but we remain consistent and continue working in our 'untwisted' setting.

Let $\varepsilon_+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\varepsilon_- = \varepsilon_+^t$, and consider for $g \in \mathbb{N}_0$ the finite dimensional vector space

$$\Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C}) = \left\{ \xi = \sum_{d=-1}^{g} \xi_d \lambda^d \mid \xi_{-1} \in \mathbb{C}\varepsilon_+,\, \xi_d = -\bar{\xi}^t_{g-1-d} \in \mathfrak{sl}_2(\mathbb{C}) \text{ for } d = -1, \ldots, g \right\}.$$

Clearly $\Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$ is a real $3g+2$-dimensional vector space and has up to isomorphism a unique norm $\|\cdot\|$. These Laurent polynomials define smooth mappings from $\lambda \in \mathbb{S}^1$ into $\mathfrak{sl}_2(\mathbb{C})$. Note that $\sqrt{\lambda} \mapsto \lambda^{\frac{1-g}{2}}\xi$ belongs to the loop Lie algebra $\Lambda\mathfrak{su}_2$ of the loop Lie group $\Lambda\mathrm{SU}_2$. For the resulting solution of the sinh-Gordon equation to be of finite type, we need in addition the conditions $\operatorname{tr}(\xi_{-1}\varepsilon_-) \neq 0 \neq \operatorname{tr}(\xi_0\varepsilon_+)$. These conditions ensure that ξ_{-1} and the lower left entry of ξ_0 do not vanish, and therefore that $\xi_{-1} + \operatorname{tr}(\xi_0\varepsilon_+)\,\epsilon_-$ is semisimple. This is the same as semisimplicity of the leading order term in the twisted setting. We thus define

$$\Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times = \left\{ \xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C}) \mid \operatorname{tr}(\xi_{-1}\varepsilon_-) \neq 0 \neq \operatorname{tr}(\xi_0\varepsilon_+) \right\}.$$

By the Symes method [65], elucidated by Burstall and Pedit [11, 12], the extended framing $F_\lambda : \mathbb{R}^2 \to \Lambda\mathrm{SU}_2$ of a CMC immersion of finite type is given by the unitary factor of the Iwasawa decomposition of

$$\exp(z\,\xi) = F_\lambda\, B \tag{2.1}$$

for some $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ with $g \in \mathbb{N}_0$. Due to Pressly and Segal [54], the Iwasawa decomposition is a diffeomorphism between the loop group $\Lambda\mathrm{SL}_2(\mathbb{C})$ of $\mathrm{SL}_2(\mathbb{C})$ into point wise products of elements of $\Lambda\mathrm{SU}_2$ with elements of the loop group $\Lambda^+\mathrm{SL}_2(\mathbb{C})$ of holomorphic maps from $\lambda \in \mathbb{D}$ to $\mathrm{SL}_2(\mathbb{C})$, which take at $\lambda = 0$ values in the subgroup of $\mathrm{SL}_2(\mathbb{C})$ of upper-triangular matrices with positive real diagonal entries. For every $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$ there exists a unique $\alpha(\xi) \in \Omega^1(\mathbb{R}^2,\, \Lambda^1_{-1}\mathfrak{sl}_2(\mathbb{C}))$, such that $\xi dz - \alpha(\xi)$ takes values in the Lie algebra of $\Lambda^+\mathrm{SL}_2(\mathbb{C})$ of the right hand factor in the Iwasawa decomposition (2.1).

A **polynomial Killing field** is a map $\zeta : \mathbb{R}^2 \to \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$ which solves

$$d\zeta = [\,\zeta,\, \alpha(\zeta)\,] \quad \text{with} \quad \zeta(0) = \xi. \tag{2.2}$$

For each **initial value** $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$, there exists a unique polynomial Killing field given by

$$\zeta = B\xi B^{-1} = F_\lambda^{-1}\xi F_\lambda \quad \text{with } F_\lambda \text{ and } B \text{ as in (2.1).} \tag{2.3}$$

For $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ with $\operatorname{tr}(\xi_{-1}\varepsilon_-) \in \mathbb{R}^+ i$ and $\operatorname{tr}(\xi_0\varepsilon_+) \in \mathbb{R}^+ i$ the corresponding $\Lambda^1_{-1}\mathfrak{sl}_2(\mathbb{C})$-valued 1-form $\alpha(\zeta)$ is the α as in (1.10) for that particular solution u of the sinh-Gordon equation corresponding to the extended frame F_λ of (2.1). For general initial values $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ the leading term $\zeta_{-1} + \operatorname{tr}(\zeta_0\varepsilon_+)\,\epsilon_-$ of the corresponding polynomial Killing field does not depend on the surface parameter z. The corresponding $\alpha(\zeta)$ differs from (1.10) by multiplication of λ and dz with constant unimodular complex numbers. Given a polynomial Killing field ζ, we set the initial value $\xi = \zeta|_{z=0}$ in (2.1). Thus ζ, or the initial value ξ, gives rise to an extended frame, and thus to an associated family.

Definition 2.1. *A solution of the* sinh-*Gordon equation is called a* **finite type** *solution if and only if it corresponds to a polynomial Killing field* $\zeta : \mathbb{R}^2 \to \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ *with* $g \in \mathbb{N}_0$.

2.2. **Roots of polynomial Killing fields.** If an initial value ξ has a root at some $\lambda = \alpha \in \mathbb{C}^\times$, then the corresponding polynomial Killing field has a root at the same λ for all $z \in \mathbb{C}$. In this case we may reduce the order of ξ and ζ without changing the corresponding extended frame F (2.1). The following polynomials transform under $\lambda \mapsto \bar{\lambda}^{-1}$ as

$$p(\lambda) = \begin{cases} i(\sqrt{\bar{\alpha}}\lambda - \sqrt{\alpha}) & \text{for } \alpha\bar{\alpha} = 1 \\ (\lambda - \alpha)(1 - \bar{\alpha}\lambda) & \text{for all } \alpha \in \mathbb{C} \end{cases} \qquad \overline{\lambda^{\deg(p)} p\left(\bar{\lambda}^{-1}\right)} = p(\lambda). \tag{2.4}$$

If the polynomial Killing field ζ with initial value $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ has a simple root at $\lambda = \alpha \in \mathbb{C}^\times$, then ζ/p does not vanish at α and is the polynomial Killing field with initial value $\xi/p \in \Lambda^{g-\deg(p)}_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$. Furthermore, obviously ζ and ζ/p commute, and we next show that both polynomial Killing fields ζ and ζ/p give rise to the same extended frame F_λ (2.1).

Proposition 2.2. *If a polynomial Killing field ζ with initial value $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$ has zeroes in $\lambda \in \mathbb{C}^\times$, then there is a polynomial $p(\lambda)$, such that the following two conditions hold:*

(i) *ζ/p is the polynomial Killing field with initial value $\xi/p \in \Lambda^{g-\deg p}_{-1}\mathfrak{sl}_2(\mathbb{C})$, which gives rise to the same associated family as ζ.*
(ii) *ζ/p has no zeroes in $\lambda \in \mathbb{C}^\times$.*

Proof. An appropriate Möbius transformation (3.1) transforms any root $\alpha \in \mathbb{C}^\times$ into a negative root. For such negative roots the corresponding initial values ξ and ξ/p are related by multiplication with a polynomial with respect to λ with positive coefficients. In the Iwasawa decomposition (2.1) this factor is absorbed in B. Hence the corresponding extended frames coincide, which proves (i). Repeating this procedure for every root $\lambda \in \mathbb{C}^\times$ ensures (ii). □

Hence amongst all polynomial Killing fields that give rise to a particular CMC surface of finite type there is one of smallest possible degree (without adding further poles), and we say that such a polynomial Killing field has *minimal degree*. A polynomial Killing field has minimal degree if and only if it has neither roots nor poles in $\lambda \in \mathbb{C}^\times$. We summarize two results by Burstall and Pedit [11, 12]. The first part is a variant of Theorem 4.3 in [11], the second part follows immediately from results in [12].

Theorem 2.3. *(i) A CMC immersion $f : \mathbb{R}^2 \to \mathbb{S}^3$ is of finite type if and only if there exists a polynomial Killing field ζ with initial value $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ such that the map F_λ obtained from* (2.1) *is an extended frame of f.*

(ii) In particular there exists a unique polynomial Killing field of minimal degree that gives rise to f. Thus we have a smooth 1-1 correspondence between the set of CMC immersions of finite type and the set of polynomial Killing fields without zeroes.

Proof. Point (i) is a reformulation of Theorem 4.3 in [11]. (ii) We briefly outline how to prove the existence and uniqueness of a minimal element.

If the initial value ξ gives rise to f, then the corresponding polynomial Killing field ζ can be modified according to Proposition 2.2 so that $\tilde{\zeta}$ is of minimal degree, and still giving rise to f. Hence there exists a polynomial of least degree giving rise to f.

For the uniqueness, assume we have two initial values ξ, $\tilde{\xi}$ of least degree g both giving rise to f. Putting Proposition 3.3 and Corollary 3.8 in [12] together gives: Two finite type initial values give rise to the same associated family if and only if they commute and have equal residues. Since the residues coincide and both ξ, $\tilde{\xi}$ are of minimal degree, we conclude that $\xi \equiv \tilde{\xi}$. The unique minimal polynomial Killing field is thus $\zeta = F_\lambda^{-1}\xi\, F_\lambda$.

Since the Iwasawa factorization is a diffeomorphism, and all other operations involved in obtaining an extended frame from ζ are smooth, the resulting CMC surface depends smoothly on the entries of ζ. □

2.3. **Spectral curves I.** Due to (2.3) the characteristic equation

$$\det(\nu\, \mathbb{1} - \zeta) = \nu^2 + \det(\zeta) = 0 \tag{2.5}$$

of a polynomial Killing field ζ with initial value $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$ does not depend on $z \in \mathbb{C}$ and agrees with the characteristic equation of the initial value ξ. If $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$ then we may write $-\det\xi = \lambda^{-1}a$ for a polynomial a of degree at most $2g$ which satisfies the reality condition

$$\lambda^{2g}\overline{a(\bar\lambda^{-1})} = -a(\lambda). \tag{2.6}$$

Consequently the hyperelliptic curve has three involutions

$$\sigma : (\lambda, \nu) \mapsto (\lambda, -\nu) \qquad \rho : (\lambda, \nu) \mapsto (\bar\lambda^{-1}, \bar\lambda^{-g}\bar\nu) \qquad \eta : (\lambda, \nu) \mapsto (\bar\lambda^{-1}, -\bar\lambda^{-g}\bar\nu) \tag{2.7}$$

If a has $2g$ pairwise distinct roots, then $\nu^2 = \lambda^{-1}a(\lambda)$ is a spectral curve of genus g of a not necessarily periodic solution of the sinh-Gordon equation. The genus g is called the **spectral genus**.

Lemma 2.4. *Let a be a polynomial of degree $2g$ satisfying* (2.6)*. Then the* **isospectral set**

$$\mathcal{K}_a = \left\{\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C}) \mid \det\xi(\lambda) = -\lambda^{-1}a(\lambda)\right\}$$

is compact. Furthermore, if the $2g$ roots of a are pair wise distinct, then $\mathcal{K}_a \cong \left(\mathbb{S}^1\right)^g$.

Proof. For the compactness it suffices to show that all Laurent coefficients of a $\xi \in \mathcal{K}_a$ are bounded, since $\mathcal{K}_a$ is a closed subset of the $(3g+2)$-dimensional vector space $\Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$. For $d = (1-g)/2$ the product $\lambda^d\xi$ is skew hermitian on $|\lambda| = 1$. The negative determinant of traceless skew hermitian 2×2 matrices is the square of a norm. Hence for all $\xi \in \mathcal{K}_a$ the Laurent polynomial of $\lambda^d\xi(\lambda)$ with respect to $\sqrt{\lambda}$ is bounded on $|\lambda| = 1$. Thus the Laurent coefficients are bounded.

If a has $2g$ pairwise distinct roots, then ξ has no roots since at all roots of ξ, the determinant $\det\xi$ has a root of order two. If α is a root of a, then $\det\xi(\alpha)$ vanishes and $\xi(\alpha)$ is nilpotent. For a nonzero nilpotent 2×2-matrix $\xi(\alpha)$ there exists a 2×2-matrix Q such that $\xi(\alpha) = [Q,\, \xi(\alpha)]$. Hence for every root α of a, there exists a 2×2-matrix Q, such that

$$\dot\xi(\lambda) = \frac{\xi^{-1}\det\xi + [Q,\,\xi]}{\lambda-\alpha} = \frac{-\xi + [Q,\,\xi]}{\lambda - \alpha}$$

has no pole at $\lambda = \alpha$. The corresponding derivative of $a = -\lambda\det(\xi)$ is equal to $\dot a = \frac{2a}{\lambda-\alpha}$. Furthermore, $\lambda\dot\xi$ is polynomial with respect to λ of degree g. Two appropriate linear combinations with the analogous tangent element at the root $\bar\alpha^{-1}$ of ξ change the roots α and $\bar\alpha^{-1}$ and fixes all other roots of a and respects the reality condition of $\Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$. These two linear combinations belong to the tangent space of $\Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$. Hence the derivatives of all the coefficients of a as functions on $\Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$ are non-zero at all $\xi \in \mathcal{K}_a$. By the implicit function theorem this set is therefore a g-dimensional submanifold. The corresponding eigenspaces of ξ depend holomorphically on the solutions (λ, ν) of (2.5) and define a holomorphic line bundle on the spectral curve. These **eigenbundles** have degree $g+1$, they are non-special in the sense that they have no holomorphic sections vanishing at one of the points at $\lambda = 0$ or $\lambda = \infty$, and finally they obey some reality condition. Vice versa, all holomorphic line bundles obeying these three conditions correspond to one $\xi \in \mathcal{K}_a$ (see McIntosh [46, Section 1.4]). Hitchin has shown in [25], that the third condition implies the second condition. Therefore $\mathcal{K}_a$ can be identified with the real part of one connected component of the Picard group of the spectral curve, which is a g-dimensional torus. □

If a has multiple roots, then the real part of the Jacobian of the corresponding hyperelliptic curve still acts on $\mathcal{K}_a$, but not transitively. More precisely, in case of non-unimodular multiple roots of a the set $\mathcal{K}_a$ has a stratification, whose strata are the orbits of the action of the real part of the generalized Jacobian of the singular hyperelliptic curve defined by $\nu^2 = \lambda\, a(\lambda)$. The elements of different strata have different orders of zeroes at the multiple roots of a. In

Proposition 2.2 we have seen that all $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$ are products of $\tilde{\xi} \in \Lambda^{\tilde{g}}_{-1}\mathfrak{sl}_2(\mathbb{C})$ of lower degree $\tilde{g} < g$ with polynomials of the form (2.4).

Definition 2.5. *Every finite type solution of the* sinh-*Gordon equation corresponds to a unique polynomial Killing field ζ without zeroes and initial value $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$. The curve defined by $\nu^2 = -\det\xi$ has a unique compactification to a projective curve without singularities at $\lambda = 0$ and $\lambda = \infty$. If $\det(\xi)$ has multiple roots, then we say that the solution contains* **bubbletons**. *The arithmetic genus of this hyperelliptic curve is equal to g.*

2.4. **Bubbletons.** We briefly motivate Definition 2.5 above, and refer the reader to [8, 12, 37, 46, 64] for further details. If an initial value $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ gives rise to a CMC cylinder with extended frame F_λ and monodromy M_λ, and $\beta \in \{\lambda \in \mathbb{C} : 0 < |\lambda| < 1\}$ is a point at which $M_\beta = \pm\mathbb{1}$, we define a simple factor

$$g = \begin{pmatrix} \sqrt{\frac{\lambda-\beta}{1-\bar{\beta}\lambda}} & 0 \\ 0 & \sqrt{\frac{1-\bar{\beta}\lambda}{\lambda-\beta}} \end{pmatrix}.$$

Then for any $0 < r < |\beta|$ the dressed extended frame, obtained from the r-Iwasawa factorization [12] of gF_λ, is an extended frame of a CMC cylinder with a bubbleton. On the initial value level this dressing action corresponds to $g\,\xi\,g^{-1}$ which obviously has singularities at β, $1/\bar{\beta}$. To eliminate these, consider $\tilde{\xi} = (\lambda-\beta)(1-\bar{\beta}\,\lambda)\,g\,\xi\,g^{-1}$. If $a = -\lambda\det\xi$, then $\tilde{a} = -\lambda\det\tilde{\xi} = (\lambda-\beta)^2(1-\bar{\beta}\,\lambda)^2\,a$, so the polynomial $\tilde{a}$ of a bubbleton has double zeroes.

Lemma 2.6. *If a has multiple roots, then $\mathcal{K}^\circ_a = \{\xi \in \mathcal{K}_a \mid$ all roots of ξ are unimodular $\}$ is open and dense in $\mathcal{K}_a$. If a has no unimodular zeroes then it is a g-dimensional submanifold. If a has unimodular zeroes, then let $\tilde{a}$ denote the quotient of a by all real zeroes. Then $\mathcal{K}^\circ_a$ is the image of the multiplication with an appropriate rational function p from $\mathcal{K}^\circ_{\tilde{a}}$ to $\mathcal{K}^\circ_a$.*

Proof. Similar arguments as in the proof of Lemma 2.4 carry over to this situation. □

Corollary 2.7. *Suppose a is polynomial of degree $2g$ satisfying the reality condition* (2.6). *Assume a has precisely $2\tilde{g}$ pairwise distinct non-unimodular roots and $g-\tilde{g}$ pairs of unimodular roots of order 2. Then $\mathcal{K}_a \cong (\mathbb{S}^1)^{\tilde{g}}$.*

Proof. Since $-\det$ is the square of a norm on all skew-hermitian 2×2 matrices, all $\xi \in \mathcal{K}_a$ have a zero at the unimodular double roots of a. Let $a(\lambda) = \tilde{a}(\lambda)p^2(\lambda)$ be the corresponding decomposition of a into an $\tilde{a}$ with pairwise distinct roots and the corresponding factors (2.4). Due to Proposition 2.2 the one-to-one correspondence $\zeta \leftrightarrow \tilde{\zeta}$ between polynomial Killing fields ζ with roots and polynomial Killing fields without roots $\tilde{\zeta}$ induces an isomorphism $\mathcal{K}_a \simeq \mathcal{K}_{\tilde{a}}$. The assertion now follows from Lemma 2.4. □

2.5. **Spectral curves II.** We also utilize the description of finite type CMC surfaces in $\mathbb{S}^3$ via spectral curves due to Hitchin [25], and relate this to our previous definition of spectral curves due to Bobenko [6]. While Hitchin defines the spectral curve as the characteristic equation for the holonomy of a loop of flat connections, Bobenko defines the spectral curve as the characteristic equation of a polynomial Killing field. We shall use both of these descriptions, and briefly recall their equivalence: Due to (1.12), the monodromy $\mathbb{C}^* \to \mathrm{SL}_2(\mathbb{C})$, $\lambda \mapsto M_\lambda$ is a holomorphic map with essential singularities at $\lambda = 0, \infty$. By construction the monodromy takes values in SU_2 for $|\lambda| = 1$. The monodromy depends on the choice of base point, but its conjugacy class and hence eigenvalues μ_λ, μ_λ^{-1} do not. With $\Delta(\lambda) = \mathrm{tr}(M_\lambda)$ the characteristic equation reads

$$\mu_\lambda^2 - \Delta(\lambda)\,\mu_\lambda + 1 = 0\,. \tag{2.8}$$

The set of solutions $(\lambda,\,\mu) \in \mathbb{C}^2$ of (2.8) yields another definition of the **spectral curve** of periodic (not necessarily finite type) solutions of the sinh-Gordon equation. Moreover, the eigenspace of M_λ depends holomorphically on $(\lambda,\,\mu)$ and defines the **eigenbundle** on the spectral curve. Let us compare this with the previous definition of a spectral curve of periodic finite type solutions of the sinh-Gordon equations. Let ζ be a polynomial Killing field with initial value $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$, with period τ so that $\zeta(p+\tau) = \zeta(p)$ for all $p \in \mathbb{R}^2$. Then also the corresponding $\alpha(\zeta)$ is τ-periodic. Let $dF_\lambda = F_\lambda \alpha(\zeta)$, $F_\lambda(0) = \mathbb{1}$ and $M_\lambda = F_\lambda(\tau)$ be the monodromy with respect to τ. Then for $z = 0$ we have $\xi = \zeta(0) = \zeta(\tau) = F_\lambda^{-1}(\tau)\,\xi\, F_\lambda(\tau) = M_\lambda^{-1}\xi\, M_\lambda$ and thus

$$[\, M_\lambda,\, \xi\,] = 0\,.$$

All eigenvalues of holomorphic 2×2 matrix valued functions depending on $\lambda \in \mathbb{CP}^1$ and commuting point wise with M_λ or ξ define the sheaf of holomorphic functions of the spectral curve. Hence the eigenvalues of ξ and M_λ are different functions on the same Riemann surface. Furthermore, on this common spectral curve the eigenspaces of M_λ and ξ coincide point-wise. Consequently the holomorphic eigenbundles of M_λ and ξ coincide.

Proposition 2.8. *A finite type solution of the* sinh-*Gordon equation is periodic if and only if*

(i) *There exists a meromorphic differential $d\ln\mu$ on the spectral curve with second order poles without residues at the two points $\lambda = 0$ and $\lambda = \infty$.*
(ii) *This differential is the logarithmic derivative of a function μ on the spectral curve which transforms under the involutions* (2.7) *as $\sigma^*\mu = \mu^{-1}$, $\rho^*\mu = \bar{\mu}^{-1}$ and $\eta^*\mu = \bar{\mu}$.*

Conversely, a periodic solutions of the sinh-*Gordon equation is of finite type if and only if the monodromy* (1.12) *fails at only finitely many points $\lambda \in \mathbb{C}^\times$ to be semisimple.*

Proof. Due to Krichever [40], the translations by $z \in \mathbb{C}$ act on the eigenbundle by the tensor product with a one-dimensional subgroup of the Picard group. In Sections 1.4-1.7 McIntosh [46] describes this Krichever construction for finite type solutions of the sinh-Gordon equation. The line bundle corresponding to $\tau \in \mathbb{C}$ is trivial if and only if there exists a non-vanishing holomorphic function μ on the compactified spectral curve with essential singularities at $\lambda = 0$ and $\lambda = \infty$, whose logarithm has a first order pole at $\lambda = 0$ and $\lambda = \infty$ with singular part equal to $\tau/\sqrt{\lambda}$ and $\bar{\tau}\sqrt{\lambda}$. This implies the characterization of periodic finite type solutions.

At all simple roots of $\Delta^2 - 4$ the monodromy (1.12) cannot be semisimple. Furthermore, at a double root of $\Delta^2 - 4$ the monodromy fails to be semisimple, if and only if it is dressed by a simple factor and contains a corresponding bubbleton. An asymptotic analysis shows that there can exists at most finitely many roots of $\Delta^2 - 4$ of order larger than two. $\square$

Pinkall and Sterling [51], and independently Hitchin [25] proved that doubly periodic solutions of the sinh-Gordon are of **finite type**. Thus all metrics of CMC tori are of finite type. We enlarge this class by relaxing one period, and make the following

Definition 2.9. *The* CMC *cylinders with constant Hopf differential and whose metric is a periodic solution of finite type of the* sinh-*Gordon will be called* CMC **cylinders of finite type**.

2.6. **Examples.** We compute some examples of initial values, polynomial Killing fields and extended frames for spheres, and spectral genus $g = 0,\,1$ surfaces. Formulas for all finite type surfaces in terms of theta-functions are given by Bobenko [5].

2.6.1. *Spheres.* We start with a discussion of spheres. Since the Hopf differential vanishes identically, spheres constitute a degenerate case since their conformal factor is a solution to

the Liouville equation rather than the sinh-Gordon equation, a fact also reflected in the initial value which does not satisfy the semi-simplicity condition. Consider

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z\,dz - u_{\bar z}\,d\bar z & 2\lambda^{-1}e^u\,dz \\ -2\lambda\,e^u d\bar z & -u_z\,dz + u_{\bar z}\,d\bar z \end{pmatrix}. \tag{2.9}$$

Then $2\,d\alpha_\lambda + [\,\alpha_\lambda \wedge \alpha_\lambda\,] = 0$ if and only if u solves the Liouville equation $\partial\bar\partial\, u + e^{2u} = 0$. The solution is $u(z,\,\bar z) = -\log(1+z\bar z)$. Plugging this into the α_λ in (2.9), and solving $dF_\lambda = F_\lambda\,\alpha_\lambda$, $F_\lambda(0) = \mathbb{1}$ gives the extended framing

$$F_\lambda = \frac{1}{\sqrt{1+z\bar z}}\begin{pmatrix} 1 & z\lambda^{-1} \\ -\lambda\bar z & 1\end{pmatrix}.$$

Using the initial value $\xi = \lambda^{-1}\varepsilon_+ - \lambda\,\varepsilon_-$, the corresponding polynomial Killing field is

$$\zeta = \frac{1}{1+z\bar z}\begin{pmatrix} z-\bar z & \lambda^{-1}(1+z^2) \\ -\lambda\,(1+\bar z^2) & \bar z - z\end{pmatrix}.$$

2.6.2. *Flat cylinders.* We next discuss flat CMC surfaces of revolution in $\mathbb{S}^3$, and compute the closing conditions for the Clifford torus.

Proposition 2.10. *After a possible isometry, for any flat surface $f:\mathbb{R}^2 \to \mathbb{S}^3$ with constant mean curvature H, there exists a $t_0 \in \mathbb{R}$ such that $H = \cot(2t_0)$, and with $\lambda_0 = e^{it_0}$ we have*

$$f = F_{\lambda_0^{-1}}\,F_{\lambda_0}^{-1} \quad \textit{with} \quad F_\lambda = \exp\left(\frac{i}{2}\begin{pmatrix} 0 & z\lambda^{-1}+\bar z \\ z+\bar z\lambda & 0\end{pmatrix}\right). \tag{2.10}$$

Proof. Solving $dF_\lambda = F_\lambda\alpha_\lambda$ with $u \equiv 0$ in α_λ of (1.10) proves that after a possible isometry, any flat constant mean curvature immersion is framed by such F_λ as in (2.10). Hence there exist distinct $\lambda_0,\,\lambda_1 \in \mathbb{S}^1$, and a frame of the given form such that after a possible isometry $f = F_{\lambda_1}\,F_{\lambda_0}^{-1}$. By (1.9), the two distinct unimodular numbers $\lambda_j = e^{2it_j}$, $j = 0,\,1$ must be chosen so that for the constant mean curvature H of f we have $H = \cot(t_0 - t_1)$. Now $\lambda_0,\,\lambda_1$ are determined only up to a phase, and since rotations in the λ-plane correspond to rotations in the z-plane via a unitary gauge, we may adjust the phase so that $\lambda_1 = \lambda_0^{-1}$. $\square$

For such spectral genus zero surfaces, the polynomial Killing field is constant and equal to the initial value $\xi = \frac{i}{2}\left((\lambda^{-1}+1)\,\varepsilon_+ + (1+\lambda)\,\varepsilon_-\right)$. The factor 1/2 is a consequence of the choices made in Theorem 1.1. For $\lambda_0 = i$ and $\lambda_1 = -i$ we obtain a minimal flat surface. We determine the simple periods such that the restriction to a fundamental domain gives the **Clifford torus**: The eigenvalues of $F_\lambda(z)$ are of the form $\exp(\pm\mu)$ with $\mu(z,\,\lambda) = \frac{i}{2}(z\,\lambda^{-1/2} + \bar z\,\lambda^{1/2})$. Simple periods are numbers $\omega_1,\,\omega_2 \in \mathbb{C}$ with smallest possible modulus satisfying $\mu(\omega_1,\,\pm i) = \pi i$ and $\mu(\omega_2,\,\pm i) = \pm\pi i$, and compute to $\omega_1 = \pi\sqrt{2}$ and $\omega_2 = \pi i\sqrt{2}$.

2.6.3. *Delaunay surfaces.* For $a,\,b \in \mathbb{R}$ the 1-parameter family of conformal metrics of Delaunay surfaces $v^2(x)(dx^2+dy^2)$ is given by the Jacobian elliptic functions $v(x) = 2b\,\mathrm{dn}(2bx\,|\,1-a^2/b^2)$. For the derivation of this conformal factor from the Gauss equation in the rotational case we refer to [10, 60]. Note that in the limiting cases $a = \pm b$ we have $v \equiv 1$, which is the flat case, while when $a = 0$ we have $v(x) = \mathrm{sech}(x)$. The initial value [35, 60] is $\xi = (a\lambda^{-1}+b)\,i\varepsilon_+ + (b+a\lambda)\,i\varepsilon_-$, and the polynomial Killing field computes to

$$\zeta = i\begin{pmatrix} -\frac{v'(x)}{2v(x)} & \frac{2ab\lambda^{-1}}{v(x)} + \frac{v(x)}{2} \\ \frac{2ab\lambda}{v(x)} + \frac{v(x)}{2} & \frac{v'(x)}{2v(x)}\end{pmatrix}.$$

3. Deformation of spectral data

In this section we describe the spectral curves of periodic finite type solutions of the sinh-Gordon equation by two polynomials a and b. The first polynomial defines the hyperelliptic curve and the second polynomial the meromorphic differential $d\ln\mu$ on this curve. Not all polynomials a and b correspond to spectral curves of periodic solutions of the sinh-Gordon equation. In order to describe the subsets of all such a and b, we derive vector fields on the space of coefficients of a and b, which leave these subsets invariant. We will be using the usual spectral parameter λ in which $\lambda = 0,\,\infty$ are singularities, as well as the transformed spectral parameter

$$\kappa = i\,\frac{1-\lambda}{1+\lambda}\,.$$

(Hence $\lambda = (i-\kappa)/(i+\kappa)$.) The part of the spectral curve over $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$ corresponds to real κ. These parameters are fixed only up to Möbius transformations

$$\lambda \mapsto e^{2i\varphi}\lambda \qquad\qquad \kappa \mapsto \frac{\sin\varphi + \kappa\cos\varphi}{\cos\varphi - \kappa\sin\varphi} \tag{3.1}$$

This degree of freedom allows us to assume that no branch point or otherwise significant point, which we shall introduce later (for example a zero of $d\ln\mu$), lies at $\kappa = \infty$. The spectral curve is then a hyperelliptic surface which we describe with the equation

$$\nu^2 = (\kappa^2+1)\,a(\kappa)\,. \tag{3.2}$$

Here a is a real polynomial of degree $2g$ which has highest coefficient equal to one, and which is non-negative for $\kappa \in \mathbb{R}$. Thus a only possesses real roots of even order.

In the following we will consider finite type CMC cylinders in $\mathbb{S}^3$. These possess a monodromy (1.12) whose eigenvalue μ is a holomorphic function on the spectral curve, and has essential singularities at $\kappa = \pm i$. Then $d\ln\mu$ is an abelian differential of the second kind of the form

$$d\ln\mu = 2\pi i\frac{b(\kappa)\,d\kappa}{(\kappa^2+1)\,\nu}\,, \tag{3.3}$$

where b is a real polynomial of degree $g+1$.

As a consequence of the work of Bobenko [5, 6], our starting point is the definition of what we call the spectral data of a CMC cylinder of finite type in the 3-sphere.

Definition 3.1. *Let a be a real polynomial of degree $2g$ with highest coefficient equal to one, and let b be a real polynomial of degree $g+1$, and $\kappa_0,\,\kappa_1 \in \mathbb{R}$ two* **marked points**.

The **spectral data** *of a CMC cylinder of finite type in $\mathbb{S}^3$ with mean curvature*

$$H = \frac{1+\kappa_0\kappa_1}{\kappa_0-\kappa_1} \tag{3.4}$$

consists of a quadruple $(a,\,b,\,\kappa_0,\,\kappa_1)$ with the following properties:

(A) $a(\kappa) \geq 0$ *for* $\kappa \in \mathbb{R}$.
(B) *On the hyperelliptic surface $\nu^2 = (\kappa^2+1)\,a(\kappa)$ there is a single valued holomorphic function μ with essential singularities at $\kappa = \pm i$ with logarithmic differential* (3.3), *that transforms under the three involutions*

$$\sigma : (\kappa,\,\nu) \mapsto (\kappa,\,-\nu), \qquad \rho : (\kappa,\,\nu) \mapsto (\bar\kappa,\,\bar\nu), \qquad \eta : (\kappa,\,\nu) \mapsto (\bar\kappa,\,-\bar\nu),$$

as $\sigma^*\mu = \mu^{-1}$, $\rho^*\mu = \bar\mu^{-1}$ and $\eta^*\mu = \bar\mu$.
(C) $\mu(\kappa_0) = \mu(\kappa_1) = \pm 1$.

We call the fixed point set of ρ the **real part**. While a Möbius transformation (3.1) of the parameter κ changes the spectral data $(a,\, b,\, \kappa_0,\, \kappa_1)$, it changes neither the corresponding periodic solutions of the sinh-Gordon equation nor the corresponding CMC cylinders in $\mathbb{S}^3$. Hence the moduli space of spectral data is the set of equivalence classes of spectral data up to the action on the spectral data induced by (3.1), prompting the following

Definition 3.2. (i) *For all $g \in \mathbb{N}_0$ let $\hat{\mathcal{M}}_g$ be the space of equivalence classes of spectral data $(a,\, b)$ obeying conditions* (A)-(B) *up to the action of* (3.1) *on* (a, b).

(ii) *For all $g \in \mathbb{N}_0$ let $\mathcal{M}_g$ be the space of equivalence classes of spectral data $(a,\, b,\, \kappa_0,\, \kappa_1)$ obeying conditions* (A)-(C) *up to the action of* (3.1) *on* $(a, b, \kappa_0, \kappa_1)$.

Thus $\hat{\mathcal{M}}_g$ is the moduli space of spectral data of periodic solutions of the sinh-Gordon equation of arithmetic genus g, and $\mathcal{M}_g$ is the moduli space of spectral data of finite type CMC cylinders in $\mathbb{S}^3$.

We now derive vector fields on open sets of spectral data $\{(a,\, b,\, \kappa_0,\, \kappa_1)\}$ and show that their integral curves are differentiable families of spectral data of periodic finite type solutions of the sinh-Gordon equation. We parameterize such families by one or more real parameters, which we will denote by t. From condition (B) in Definition 3.1 we conclude that $\partial_t \ln \mu$ is meromorphic on the corresponding family of spectral curves. If we view these functions locally in terms of κ and t, then $\partial_t \ln \mu$ can only have poles at the branch points, or equivalently at the zeroes of a, and at $\kappa = \pm i$. If we assume that for such a family of spectral curves the genus g is constant, then $\partial_t \ln \mu$ can at most have poles of first order at simple roots of a. In general we have

$$\partial_t \ln \mu = \tfrac{2\pi i}{\nu}\, c(\kappa) \tag{3.5}$$

with a real polynomial c of degree at most $g+1$.

To compute the corresponding vector field on the space of spectral data we view μ locally as a function of the parameters κ and t. Differentiating (3.3) and (3.5) gives

$$\partial^2_{t\kappa} \ln \mu = 2\pi i \frac{2\dot{b}a - b\dot{a}}{2\nu^3}\,, \quad \partial^2_{\kappa t} \ln \mu = 2\pi i \frac{2(\kappa^2+1)ac' - 2\kappa ac - (\kappa^2+1)a'c}{2\nu^3}\,.$$

Second partial derivatives commute if and only if

$$2\dot{b}a - b\dot{a} = 2(\kappa^2+1)ac' - 2\kappa ac - (\kappa^2+1)a'c\,. \tag{3.6}$$

The highest coefficient on the right hand side vanishes, so both sides are polynomials of at most degree $3g+1$. As the highest coefficient of a does not depend on t we conclude that $\dot{a}$ is a real polynomial of degree $2g-1$, and $\dot{b}$ a real polynomial of degree $g+1$. Thus we have to determine $3g+2$ real coefficients. In case a and b have no common roots, equation (3.6) uniquely determines the values of $\dot{a}$ at the roots of a and the values of $\dot{b}$ at the roots of b. Since the highest coefficient on the right hand side depends only on the highest coefficient of $\dot{b}$, in this case (3.6) uniquely determines a tangent vector on the space of spectral data of periodic finite type solutions of the sinh-Gordon equation. By defining such polynomials c we obtain vector fields on the space of real polynomials a of degree $2g$ and highest coefficient one and real polynomials b of degree $g+1$.

For spectral data of CMC cylinders in $\mathbb{S}^3$ we have to deform in addition to the polynomials a and b the two marked points, such that the closing condition (C) of Definition 3.1 is preserved. As long as $\kappa_0 \neq \kappa_1$, and thus $|H| < \infty$, we preserve the closing condition if $\partial_t \ln \mu(\kappa_j(t),\, t) = 0$, which holds precisely when $\partial_{\kappa_j} \ln \mu(\kappa_j(t),\, t)\, \partial_t \kappa_j + \partial_t \ln \mu(\kappa_j(t),\, t) = 0$. Using equations (3.3) and (3.5), the closing conditions are therefore preserved if and only if

$$\dot{\kappa}_j = -\frac{(\kappa_j^2+1)\, c(\kappa_j)}{b(\kappa_j)}\,. \tag{3.7}$$

The equations (3.6) and (3.7) define rational vector fields on the space of spectral data $(a, b, \kappa_0, \kappa_1)$ not necessarily obeying conditions (A)-(C) of Definition 3.1

Theorem 3.3. (i) *Let U be an open subset of spectral data $(a,\, b)$, with a, b having no common roots. Let c be a smooth function from U to the real polynomials of degree $g+1$. Then equations* (3.6) *define a smooth vector field on U. The corresponding flow leaves invariant the subset of spectral data obeying conditions* (A)-(B) *of Definition 3.1.*

(ii) *Let U be an open subset of spectral data $(a,\, b,\, \kappa_0,\, \kappa_1)$, with a, b having no common roots, and real $\kappa_0 \neq \kappa_1$ Let c be a smooth function from U to the real polynomials of degree $g+1$. Then equations* (3.6) *and* (3.7) *define a smooth vector field on U. The corresponding flow leaves invariant the subset of spectral data obeying conditions* (A)-(C) *of Definition 3.1.*

Proof. We only prove (i). The proof of (ii) is similar. The solutions $\dot{a}$ and $\dot{b}$ of equation (3.6) are rational expressions of the coefficients of a, b and c. If a and b have no common roots, then the Taylor coefficients of $\dot{a}$ and $\dot{b}$ at the roots of a and b up to the order of the roots minus one, respectively, and the highest coefficient of $\dot{b}$ are uniquely determined by equation (3.6). Hence, in this case the denominators of the rational expressions for $\dot{a}$ and $\dot{b}$ do not vanish. Hence for smooth c the corresponding $\dot{a}$ and $\dot{b}$ are smooth too.

Due to (3.5) $\partial_t \ln \mu$ is a meromorphic function on the hyperelliptic curve. Hence the periods of the meromorphic differential $d \ln \mu$ do not depend on t. The transformation rules of μ under σ, ρ and η are preserved under the flows of the vector field corresponding to c. Hence the integrals of $d \ln \mu$ along any smooth path from one root of a to another root of a is preserved too. This implies that the subset of spectral data (a, b), which determine by (3.3) a single valued function μ with $\sigma^* \mu = \mu^{-1}$, $\rho^* \mu = \bar{\mu}^{-1}$ and $\eta^* \mu = \bar{\mu}$ is preserved under this flow. □

Remark 3.4. *The space of real hyperelliptic curves of genus g is up to Möbius transformations is $2g-1$ dimensional. All such curves correspond to real solutions of the* sinh-*Gordon equation. The subset $\hat{\mathcal{M}}_g$ of curves corresponding to periodic solutions has codimension growing with g. The space of real polynomials c of degree at most $g+1$ is $g+2$-dimensional. In case c is proportional to b, then the deformation corresponds to an infinitesimal Möbius transformation* (3.1). *Hence $\hat{\mathcal{M}}_g$ is $g+1$-dimensional.*

4. Moduli of spectral data of periodic solutions of the sinh-Gordon equation

In this section we switch to another description of the spectral data of Definition 3.1. We shall use the values of the trace Δ of the monodromy (2.8) at the roots of the derivative of Δ, as in Grinevich and Schmidt [22], to determine local parameters on the moduli of spectral curves of genus g. In doing so, and switching to the parameter κ of the previous section, we describe the twice-punctured Riemann sphere $\mathbb{CP}^1 \setminus \{\pm i\}$, which is the domain of definition of the covering map Δ, as the result of gluing infinitely many copies of $\mathbb{C}$ along cuts to be specified by the branch points and combinatorial data. The combinatorial data specifies which sheets are joined by which branch points and branch cuts. We call this combinatorial data the **gluing rules**.

The meromorphic function κ and the holomorphic function μ with essential singularities at $\kappa = \pm i$ fulfill an equation of the form

$$\mu^2 - \Delta(\kappa)\, \mu + 1 = 0\,. \tag{4.1}$$

Hence Δ is a holomorphic function

$$\Delta\, :\, \mathbb{CP}^1 \setminus \{\pm i\} \to \mathbb{C}\,, \qquad \kappa \mapsto \Delta(\kappa) \tag{4.2}$$

The curve (4.1) is hyperelliptic with hyperelliptic involution $\sigma : (\kappa, \mu) \mapsto (\kappa, \mu^{-1})$. The branch points are the odd-ordered roots of $\Delta^2 - 4$. We next characterize those maps Δ which correspond to spectral data $(a,\, b)$ described in Definition 3.1.

Let us indicate to what extent the function Δ is determined by its values at the branch points, and the gluing rules. We first recall the simpler, but essentially comparable situation of finitely sheeted coverings investigated by Hurwitz [28]. Finitely sheeted covering maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ are determined by their branch points and their gluing rules up to Möbius transformations of both copies of $\mathbb{CP}^1$. In most cases the parametrization of the image of the covering map is fixed by the values of this parameter at the branch points. By fixing the values of the parameter of the domain at some marked points, we can fix also the parameter of the domain. For some fixed parametrization of the domain and the image such a finitely sheeted covering map is a rational function. Consequently a rational function can be characterized by the values of the function at the zeroes of the derivative, some gluing rules, and some conditions on the parametrization of the domain.

Now we return to our infinitely sheeted covering map (4.2). In this article we shall be concerned only with spectral curves of finite geometric genus. In this case $\ln \mu$ extends to a meromorphic function on two neighbourhoods of $\kappa = \pm i$ with first order poles at these two points. In particular $(\ln \mu)^{-2} = -(\arccos(\Delta/2))^{-2}$ is a local parameter on $\kappa \in \mathbb{CP}^1 \setminus \{\pm i\}$ at $\kappa = \pm i$. With this local parameter the domain of (4.2) can be compactified to $\mathbb{CP}^1$. Since κ takes at the two marked points the values $\pm i$ and transforms under ρ and η as $\kappa \mapsto \bar{\kappa}$, the parameter κ is determined up to Möbius transformations (3.1).

Theorem 4.1. *The functions $(\kappa,\, \mu)$ correspond to spectral data $(a,\, b)$ obeying conditions* (A) *and* (B) *in Definition 3.1 if and only if the function Δ satisfies the following conditions:*

- (D) *All but finitely many roots of $\Delta^2 - 4$ are roots of even order.*
- (E) *A branch of the function $(\ln \mu)^{-2} = -(\arccos(\Delta/2))^{-2}$ extends to a holomorphic function on two neighbourhoods of $\kappa = \pm i$ with simple zeroes at $\kappa = \pm i$.*
- (F) *The function Δ transforms as $\Delta(\bar{\kappa}) = \bar{\Delta}(\kappa)$ and obeys $\Delta^2(\kappa) \leq 4$ for all $\kappa \in \mathbb{R}$.*

The roots of the corresponding polynomial a have to be a subset of the roots of $\Delta^2 - 4$, such that $(\Delta^2 - 4)/a$ has only roots of even order. All such real a are possible choices. Conversely, the function Δ is uniquely determined by the spectral data $(a,\, b)$.

Proof. The odd order roots of $\Delta^2 - 4$ are the branch points of the two-sheeted covering (4.1) over $\kappa \in \mathbb{CP}^1$. Hence only Δ obeying condition (D) correspond to polynomials a. Since $\kappa = \pm i$ is a branch point, condition (E) follows from condition (B). Moreover, the involutions ρ and η in condition (B) induce the involution in Condition (F).

Vice verse, Condition (F) implies that all real roots of $\Delta^2 - 4$ are roots of even order. Locally on $\kappa \in \mathbb{CP}^1 \setminus \{\pm i\}$ the function μ is holomorphic in terms of Δ and a square root of $\Delta^2 - 4$. Hence for all real a, such that $(\Delta^2 - 4)/a$ has only roots of even order, the differential $d \ln \mu$ is a meromorphic differential on the hyperelliptic curve defined by $\nu^2 = (\kappa^2 + 1)a(\kappa)$. This differential is antisymmetric with respect to the hyperelliptic involution $\sigma : (\kappa, \nu) \mapsto (\kappa, -\nu)$. Furthermore, due to condition (E) it has second order poles at $\kappa = \pm i$ without residues and no other poles. Hence it is of the form (3.3). Due to condition (F) the involution $\kappa \mapsto \bar{\kappa}$ induces on the hyperelliptic curve two antilinear involutions ρ and η, one of which denoted by η has no fixed points. The function μ transforms as $\rho^* \mu = \bar{\mu}^{-1}$ and $\eta^* \mu = \bar{\mu}$. $\square$

We shall see later that condition (E) implies (D). The function Δ defines an infinitely sheeted covering map with essential singularities at $\kappa = \pm i$. The branch points of this covering map are

the zeroes of $\Delta'(\kappa) = 0$ and thus precisely the roots of $d\ln\mu$ together with the set of singularities of the spectral curve defined by equation (4.1).

We shall see that essentially we can move all the branch points independently without destroying the periodicity. The reality condition (F) in Theorem 4.1 imposes the only restriction. Consequently the moduli space is a covering space over the parameter space of the values of Δ at all branch points. We shall see that for spectral curves of finite geometric genus all of them with the exception of finitely many are fixed. Consequently we can assume this parameter space to have finite dimension. The gluing rules of this covering are completely determined by the gluing rules of the covering (4.2). In Hurwitz [28] the analogous deformations of finitely sheeted coverings are investigated.

Lemma 4.2. *For spectral data $(a,\, b)$ of genus g obeying conditions* (A)-(B) *of Definition 3.1, for which none of the pairwise distinct roots of b are roots of a, the element $[(a,\, b)] \in \hat{\mathcal{M}}_g$ possesses an open neighbourhood in $\hat{\mathcal{M}}_g$, which is uniquely parameterized by the values of Δ at the roots of b.*

Proof. By assumption there exists for every root β_i of b a unique polynomial c_i that vanishes at every root of b except at β_i, where it attains a value such that

$$\partial_t \Delta(\beta_i) = 2\sinh(\ln(\mu(\beta_i)))\partial_t \ln\mu(\beta_i) = 1\,.$$

The corresponding vector fields in a neighbourhood U of spectral data in the moduli space commute: the map $U \to \mathbb{C}^{g+1}$, $\beta_i \mapsto \Delta(\beta_i)$ sends these vector fields to coordinate vector fields. Since these vector fields are linearly independent on U, they generate the quotient of all polynomials c modulo the polynomials b. Now the Lemma follows from Theorem 3.3. □

In the following we shall investigate how the above parametrization is affected if the roots of the polynomial b are either not all distinct, or some of the roots of b coincide with the roots of a. Recall that the local parameters are the values of Δ at the branch points of the cover $\kappa \mapsto \Delta(\kappa)$. Describing this cover by means of the branch cuts we also have a description of the moduli space as a cover of the parameter space, thus obtaining a global picture of the moduli space. Assume we are given two branch points of the covering map $\kappa \mapsto \Delta(\kappa)$ that connect a given sheet of the cover with two different sheets. If in the process of a continuous deformation one of these branch points circumvents the other branch point, then the sheets which these branch points connect permute. Hence higher order roots of b are branch points of the moduli space.

If on the other hand, a root of b coincides with a root of a, then the derivative of $\kappa \mapsto \Delta(\kappa)$ has a higher order root there and $\Delta^2(\kappa) = 4$ there. Thus a root of b coincides with a root of Δ' that arises from a singularity of the spectral curve (4.1). Hence also in this case two branch points of the covering map $\kappa \mapsto \Delta(\kappa)$ coalesce.

For any branch point of a covering map (4.2) we can choose small open neighbourhoods U and V in the domain and in the image respectively, such that the restriction $\Delta|_U$ of Δ to U is a finitely sheeted covering over V. Small movements of the branch points of $\Delta|_U$ over a closed subset $A \subset V$ do not change the restriction of Δ to the pre-image of $V \setminus A$ under $\Delta|_U$. Moreover, if we consider higher order branch points as coalescing first order branch points, we can move all these first order branch points independently over A, without changing the restrictions of Δ to the pre-image of $V \setminus A$ under $\Delta|_U$. More precisely, if β is a branch point at $\Delta_0 = \Delta(\beta)$ of order k, then $w = \sqrt[k+1]{\Delta - \Delta_0}$ is a local parameter of the covering space on an open neighbourhood of β. All polynomials $P(w)$ of degree $k+1$ with highest coefficient 1 and small lower order coefficients describe small perturbations of the covering map $w \mapsto \Delta = w^{k+1} + \Delta_0$. All of them are covering maps with k branch points, which are the zeroes of the derivative P'. The parameter w of such

coverings is determined up to Möbius transformations $w \mapsto aw + b$ with $a \in \mathbb{C}^\times, b \in \mathbb{C}$. Hence the values of Δ at the k branch points together with the gluing rules of these covering maps $w \mapsto P(w)$ determine such P uniquely up to a reparametrization $\tilde{P}(w) = P(aw + b)$ with $a \in \mathbb{C}^\times, b \in \mathbb{C}$. Furthermore, all small values of P at the k branch points are realized by such polynomials. By gluing the deformed covering $\tilde{\Delta}|_U$ along the pre-image of $V \setminus A$ under $\Delta|_U$ with the restriction of the undeformed Δ to $\mathbb{CP}^1 \setminus (\{i, -i\} \cup A)$ we obtain a deformation $\tilde{\Delta}$ of the covering Δ, as a covering map from an abstract Riemann surface without fixed parameter κ onto $\mathbb{C}$ with fixed parameter Δ.

Finite combinations of such deformations we call **local deformations**. If the movements of the branch points respect condition (F) in Theorem 4.1, then we call the corresponding deformations **real local deformations**.

In Section 3 we described deformations of spectral data by real polynomials c of degree $g + 1$. We shall calculate the corresponding deformation of (4.2). If we consider also Δ and μ as functions depending on κ and t, then we have due to (3.3) and (3.5)

$$\Delta' = 2\sinh(\ln\mu)(\ln\mu)' = 4\pi i \frac{\sinh(\ln\mu) b}{(\kappa^2+1)\nu} \qquad \dot{\Delta} = 2\sinh(\ln\mu)(\dot{\ln\mu}) = 4\pi i \frac{\sinh(\ln\mu) c}{\nu}.$$

Hence $\dot{\Delta}$ is given by

$$\dot{\Delta} = \frac{(\kappa^2+1)c}{b}\Delta'. \tag{4.3}$$

Lemma 4.3. *Local deformations preserve conditions* (D)-(E) *in Theorem 4.1. In particular, real local deformed $\tilde{\Delta}$ corresponds to a parameterized covering map* (4.2)*, but the parameter κ is determined only up to Möbius transformations* (3.1)*. Those real local deformations $\tilde{\Delta}$ of the covering map Δ corresponding to spectral data (a, b) obeying conditions* (A)-(B) *of Definition 3.1, which move only the branch points of Δ corresponding to roots of b, correspond to unique $[(\tilde{a}, \tilde{b})] \in \hat{\mathcal{M}}_g$.*

Proof. Since local deformations move only finitely many branch points, they preserve conditions (D)-(E) in Theorem 4.1. Due to condition (E) the covering space corresponding to a local deformed $\tilde{\Delta}$ can be compactified to $\mathbb{CP}^1$. A parameter κ, which takes at the two added points the values $\pm i$ and transforms under $\Delta \mapsto \bar{\Delta}$ as $\kappa \mapsto \bar{\kappa}$ is unique up to (3.1). Due to this Theorem all real local deformations $\tilde{\Delta}$ of Δ corresponding to spectral data (a, b) correspond also to spectral data $(\tilde{a}, \tilde{b})$. The branch points of $\tilde{\Delta}$ are in one-to-one correspondence with the zeroes of $d\ln\mu$ and the singularities of (4.1). Hence a real local deformation, which moves only the branch points corresponding to zeroes of b corresponds to deformed spectral data $(\tilde{a}, \tilde{b})$, which are unique up to the Möbius transformations (3.1). □

Since the space of local deformations of the covering map (4.2) are manifolds, this Lemma can be used to make $\hat{\mathcal{M}}_g$ into a real $(g + 1)$-dimensional manifold. In the sequel we shall call those real local deformations, which can be realized as combinations of continuous movements of single branch points, **continuous deformations**. It is not difficult to introduce a topology on $\mathcal{M}_g$, such that these deformations corresponds to continuous paths. In fact the space of real polynomials of fixed degree have a natural topology. As a quotient space of such spaces, $\mathcal{M}_g$ also has a natural topology.

We shall first compute the map $\kappa \mapsto \Delta(\kappa)$ for the periodic solutions of the sinh-Gordon equation of spectral genus zero: Then $a \equiv 1$ and

$$\ln\mu = 2\pi i \frac{b_0\kappa - b_1}{\nu}. \tag{4.4}$$

For the anti-linear fix point free involution η of Definition 3.1 we have $\eta^*\overline{d\ln\mu} = d\ln\mu$. As b_0 and b_1 are real we obtain

$$d\ln\mu = 2\pi i\tfrac{b_0+b_1\kappa}{(\kappa^2+1)\,\nu}\,. \tag{4.5}$$

Then

$$\Delta(\kappa) = 2\cosh(\ln\mu) = 2\cos\left(2\pi\tfrac{b_0\kappa-b_1}{\nu}\right) = 2\cos\left(\sqrt{4\pi^2\tfrac{b_0^2\kappa^2-2b_0b_1\kappa+b_1^2}{\kappa^2+1}}\right) \tag{4.6}$$

which is the composition of the two maps $\kappa\mapsto\delta = 4\pi^2\frac{b_0^2\kappa^2-2b_0b_1\kappa+b_1^2}{\kappa^2+1}$ and $\delta\mapsto\Delta = 2\cos\sqrt{\delta}$. Of the second of these maps we choose the cuts along the lines

$$\sqrt{\delta}\in\pi n + i\,\mathbb{R}\Longleftrightarrow\begin{cases}\Delta\in(-\infty,\,-2] & \text{for } n \text{ odd}\,,\\ \Delta\in[2,\,\infty) & \text{for } n \text{ even}\,.\end{cases}$$

Besides the branch points that arise for each $n\in\mathbb{Z}^\times$ we have two additional branch points of $\kappa\mapsto\delta(\kappa)$, situated at $\delta=0$ and

$$\kappa = -\tfrac{b_0}{b_1}\Longleftrightarrow\delta = 4\pi^2\tfrac{b_0^4+2b_0^2b_1^2+b_1^4}{b_0^2+b_1^2} = 4\pi^2(b_0^2+b_1^2)\,.$$

Between these latter two points we cut along the line segment joining them. The covering map $\kappa\mapsto 2\cos\sqrt{\pi^2\kappa^2/(\kappa^2+1)}$ has sheets numbered by two copies of $\mathbb{N}$. We denote these sheets by l^+ and l^- with $l\in\mathbb{N}$. In order to describe the movements of the branch points it is convenient to choose only parallel cuts on each sheet. Besides the branch cut along the real part we choose on the sheets $l^\pm$

$$\begin{array}{lll} \text{for } l\geq 1 \text{ a cut along } \Delta\in(-1)^l2+i\mathbb{R}_0^+ & \text{connecting the sheets } l^+ \text{ and } (l+1)^+\\ \text{for } l\geq 1 \text{ a cut along } \Delta\in(-1)^l2-i\mathbb{R}_0^+ & \text{connecting the sheets } l^- \text{ and } (l+1)^-\\ \text{for } l> 1 \text{ a cut along } \Delta\in-(-1)^l2+i\mathbb{R}_0^+ & \text{connecting the sheets } l^+ \text{ and } (l-1)^+\\ \text{for } l> 1 \text{ a cut along } \Delta\in-(-1)^l2-i\mathbb{R}_0^+ & \text{connecting the sheets } l^- \text{ and } (l-1)^-\end{array} \tag{4.7}$$

Hence besides the sheets $1^\pm$ each sheet has at $\Delta=2$ and $\Delta=-2$ a branch point. Those sheets, whose labels have exponents $+$, do not have branch points and cuts at small imaginary values of Δ. Those sheets, whose labels have exponent $-$, do not have branch points and cuts at large imaginary values of Δ. Each sheet has exactly one branch point at $\Delta=-2$. This branch point connects the sheet with another sheet, to which we pass along large circles in the Δ-plane from small imaginary parts to small imaginary parts, if the label has exponent $+$ and from large imaginary parts of Δ to large imaginary parts, if the label has exponent $-$, respectively. More precisely, for all $l\in\mathbb{N}$ the branch point at $\Delta=-2$ connects the sheet with label

$$\begin{array}{rl} (2l-1)^+ & \text{with the sheet reached by traversing large circles } 2l-1 \text{ times anti-clockwise,}\\ (2l)^+ & \text{with the sheet reached by traversing large circles } 2l-1 \text{ times clockwise,}\\ (2l-1)^- & \text{with the sheet reached by traversing large circles } 2l-1 \text{ times clockwise,}\\ (2l)^- & \text{with the sheet reached by traversing large circles } 2l-1 \text{ times anti-clockwise.}\end{array} \tag{4.8}$$

The labels of the sheets are completely determined by this rule. If we choose $b_0=1/2$ and $b_1=0$, then we have an additional real branch cut that joins the sheets 1^- and 1^+ with two additional real branch points at $\Delta=\pm2$. This is the spectral data of the standard round CMC cylinder in $\mathbb{R}^3$. The spectral data (4.6) of the standard round CMC cylinders in $\mathbb{S}^3$ are obtained by moving the real branch point at $\Delta=-2$ in $\Delta\in[-2,2]$ along the sheets $2^\pm,3^\pm,\dots$ to and fro. In doing so, the real part contains branch cuts connecting the sheets 1^+ with 1^-, 2^+ with $2^-,\dots$ and l^+ with l^-. We introduce the following class of covering maps $\kappa\mapsto\Delta(\kappa)$ connecting the sheets $(l^-)_{l\in\mathbb{N}}$ and $(l^+)_{l\in\mathbb{N}}$ at branch cuts such that the following hold:

(G) There exists an $L \in \mathbb{N}$ such that for all $l > L$ the sheets $l^{\pm}$ only have the branch points and branch cuts (4.7).

(H) Along large circles in the $\Delta \in \mathbb{C}$-plane we get the following sequence of sheets:

$$\ldots, (2l+1)^+, (2l-1)^+, \ldots, 3^+, 1^+, 2^+, 4^+, \ldots, (2l)^+, (2l+2)^+, \ldots$$

traversing anti-clockwise from small imaginary parts to small imaginary parts, and

$$\ldots, (2l+1)^-, (2l-1)^-, \ldots, 3^-, 1^-, 2^-, 4^-, \ldots, (2l)^-, (2l+2)^-, \ldots$$

traversing clockwise from large imaginary parts to large imaginary parts. Reversing the anti-clockwise and clockwise order gives the same sequences in reverse order.

(I) There exists an $L \in \mathbb{N}$ such that for $l \geq L$ the total number of all branch points that join two sheets from the set $\{1^-, 1^+, \ldots, l^-, l^+\}$ is equal to $2l$. Furthermore, $2l-2$ of the corresponding branch cuts terminate at $2l-2$ additional branch points at infinity.

(J) The configuration is invariant under $l^+ \mapsto l^-$, $l^- \mapsto l^+$, $\Delta \mapsto \bar{\Delta}$. Furthermore, the branch order of real branch points at $\Delta = \pm 2$ is odd. The fixed points of the corresponding anti-linear involution is called real part and is a branch cut along $\Delta \in [-2, 2]$ between sheets l^+ and l^-

(K) Besides the branch cuts along the real part all branch cuts run along unbounded lines parallel to the imaginary axis in direction to very large or small imaginary parts according to the superscript $\pm$ of the corresponding sheets. These branch cuts start either at branch points or at the real part.

Due to (4.8) the second condition (H) fixes the labeling of the sheets. Together with condition (G) it ensures condition (E) in Theorem 4.1. With the holomorphic coordinates of this setting the corresponding covering can be compactified to a compact Riemann surface. Hence condition (G) is the finite type condition. Condition (I) ensures that the compactified covering space has genus zero. The analogous condition for finite sheeted coverings $\mathbb{CP}^1 \to \mathbb{CP}^1$ is that the branching order is two times the number of sheets minus two [28]. Condition (J) ensures the reality condition (F) in Theorem 4.1. This endows the spectral curve (4.1) with the involutions σ, ρ and η, the last of which is without fixed points. Finally condition (K) describes a choice of the corresponding branch cuts. All branch points away from the real part are the starting point of a unique branch cut running to very large or very small imaginary parts. In general, the branch cuts, which start at the real part can be moved along the real part without changing Δ. In fact, if $\Delta_0 \in (-2, 2)$ is the starting point of a branch cut connecting the sheets l^+ and m^+, and if the sheets l^+ and l^- and the sheets m^+ and m^- are connected along a real branch cut running along some neighbourhood of $\Delta_0 \in (-2, 2)$, then this Δ_0 is no branch point. Moreover the corresponding branch cut can be moved along the real part until it reaches either on the sheets $l^{\pm}$ or on the sheets $m^{\pm}$ a real branch point of Δ without changing Δ.

Lemma 4.4. *All Δ obeying conditions* (D)-(F) *in Theorem 4.1 fulfill conditions* (G)-(J). *Moreover, the branch cuts may be chosen as described in condition* (K).

Proof. For all $r > 0$ only finitely many of the infinite branch points of $\lambda \mapsto 2\cos(\sqrt{\lambda})$ belong to $B(0, r) \subset \mathbb{C}$. Hence condition (E) of Theorem 4.1 implies condition (G). For all $n \in \mathbb{N}$ the absolute value of the derivative sin of cos is not smaller than 1 on the circles $|\ln \mu| = (n + \frac{1}{2})\pi$. Hence, due to Rouche's Theorem, the number of zeroes of Δ' in the complement of small discs around $\kappa = \pm i$ with appropriate radius is the same as the corresponding number of the map (4.6). This argument is a slight variation of the Counting Lemma 2 in Chapter 2 of Pöschel and Trubowitz [53]. Therefore condition (E) of Theorem 4.1 implies also condition (I). In particular, the values of Δ at the branch points are bounded. Now condition (H) just fixes the labeling of the sheets. Condition (F) in Theorem 4.1 implies condition (J). Due to condition (I) at all non-real branch points there ends an unbounded branch cut, which may be chosen according

to condition (K). Moreover with the exception of two real branch points all real branch points correspond to an additional unbounded branch cut starting at the real part. Condition (K) describes a choice of these unbounded branch cuts. $\square$

In order to show the converse we shall show that we can deform all Δ obeying conditions (G)-(K) into the covering (4.6) and use Lemma 4.3. It would be natural to do this with decreasing geometric genus. In Theorem 6.8 we shall concentrate on deformations decreasing the following number

$$\begin{aligned} G &= \text{geometric genus} \quad + \sum_{\text{real singularities}} \delta - \text{invariant (see e.g. [62])} \\ &= \frac{1}{2}\#\{\text{non real roots of } d\ln\mu\} + \#\{\text{real branch points of } \Delta\} - 1. \end{aligned} \tag{4.9}$$

Lemma 4.5. *Any covering map* (4.2) *obeying conditions* (G)-(K) *may be continuously deformed with decreasing* G (4.9) *within this class into a covering map* (4.6).

Proof. In a first step we deform without changing the genus any covering map (4.2) obeying the five conditions (G)-(K) into a covering map, which has only branch points of first order. Nearby branch points of order k at $\Delta = \pm 2$ any deformation of $\Delta \mp 2 = z^{k+1}$ into a polynomial $p(z)$ with at least $\frac{k}{2}$ distinct double roots deforms the branch point at $\Delta = \pm 2$ into simple branch points at $\Delta = \pm 2$. A combination of such **continuous deformations** deforms all branch points at $\Delta = \pm 2$ into simple branch points at $\Delta = \pm 2$. The movement of all other branch points at $\Delta \neq \pm 2$ in $\Delta \in \mathbb{C} \setminus \{-2, 2\}$ does not change the genus at all.

In a second step we increase the values of the real branch points at $\Delta = -2$ and decrease the values of the real branch points at $\Delta = 2$ by moving them into $\Delta \in (-2, 2)$. As a result all real singularities of (4.1) are deformed into real zeroes of $d\ln\mu$ and the geometric genus becomes equal to G. In particular, Δ takes on the real part only values in $(-2, 2)$. In the remaining steps we shall decrease the geometric genus.

In a third step we increase the values at those real branch points on $\Delta \in (-2,\, 2)$, which are on the real part local minima, and decrease the values of Δ at those real branch points which are on the real part local maxima. If two real branch points coalesce, on $\Delta \in (-2,\, 2)$, then we move them away from the real part. We may continue to shrink the real part, until only two real branch points of $\kappa \mapsto \Delta(\kappa)$ remain and converge against each other. In the limit we would obtain a spectral curve with two connected components. But we stop shortly before this happens. Consequently the sheets of the covering (4.2) divide into the two groups labeled by two copies of $\mathbb{N}$, which are joined only by a small circle between the two remaining real branch points. Since all other simple branch points do not belong to the real part, they occur in complex conjugate pairs, which can be moved in complex conjugate directions.

In a fourth step we show that we can move the complex conjugate branch points over $\Delta \in \mathbb{C} \setminus \{\pm 2\}$ in such a way that the genus becomes at most equal to one. Let L denote the minimum of all $l \in \mathbb{N}$, such that for all $k > l$ the sheets $l^{\pm}$ contain the branch points (4.7) and possibly the two remaining real branch points, but no other branch point. In the subsequent discussion we neglect these real branch points, which in this step are not moved at all. Due to condition (G) the number L is finite. Now we claim that we may inductively decrease this number L until it is equal to 0. In order to avoid branch points on vertical branch cuts we move all branch points at $\Delta \neq \pm 2$ to places with pairwise different real values of Δ. Moreover, we can achieve that the real part of Δ takes at these simple branch points at $\Delta \neq \pm 2$ pairwise different values in $(-2,\, 2)$, which simplifies the subsequent argument. Furthermore if the branch points on sheets with exponents $+$ cross the lines with real part of Δ equals to ± 2 along negative imaginary values of Δ and on sheets with exponents $-$ along positive imaginary values of Δ,

the branch points do not cross the branch cuts described in (4.7). Consequently the number L is preserved under this deformation. At $\Delta = (-1)^L 2$ the sheets with labels $(L+1)^\pm$ join a unique branch point with sheets with labels not larger than L. Let $k^\pm$ denote the labels of these sheets. These sheets have besides these branch points at $\Delta = (-1)^L 2$ only branch points connecting with sheets in $\{1^-, 1^+, \ldots, L^-, L^+\}$. Due to condition (H) they have at least one other branch point. If L is even and the sheets $k^\pm$ contain more than two branch points, then we move all branch points with the exception of those with minimal and maximal real parts of Δ starting with the smaller real parts through the vertical branch cut with the lowest real parts of Δ. If L is odd, then we start with the branch points with larger real parts of Δ and move these branch points through the branch cut with the largest real part of Δ. Finally the sheets with label $k^\pm$ contain besides the branch point at $\Delta = (-1)^L 2$ exactly one other branch point. If we move this branch point to $\Delta = -(-1)^L 2$ the number L decreases. This proves the claim. Hence we can decrease the number L, until it is equal to 0 and arrive at a covering map (4.2) that corresponds to a spectral curve of genus one.

In a fifth step we finally move one of the two remaining real branch points away from the other real branch point several times along $\Delta \in [-2, 2]$ to and fro until it eventually reaches $\Delta = 2$ at the sheet with label one. In order to do so we distinguish two cases: If the real branch points are located on sheets with odd label, then we move the branch point with larger value of Δ. If both real branch points are located on a sheet with even label, then we move the branch point with smaller value of Δ. We thus obtain a covering map (4.2) of the form (4.6) corresponding to a spectral curve of geometric genus zero. □

We remark, that it is also possible to deform all Δ obeying conditions (G)-(K) with decreasing geometric genus into Δ of geometric genus zero, but the proof is more complicated. Moreover, in Lemma 7.2 and Lemma 7.3 we shall deform those Δ corresponding to finite type CMC cylinders, and again decrease G (4.9) instead of the geometric genus.

Theorem 4.6. *All spectral curves of real finite type periodic solutions of the* sinh-*Gordon equation correspond uniquely to covering maps that satisfy conditions* (G)-(K) *above. The geometric genus $g+1$ of the corresponding spectral curve is equal to the sum of weighted branch orders with the following weights:*

$$\textit{weight} = \begin{cases} \textit{branch order} & \textit{at } \Delta \neq \pm 2, \\ \frac{1}{2}(\textit{branch order}) & \textit{at } \Delta = \pm 2 \quad \textit{for even branch order}, \\ \frac{1}{2}(\textit{branch order} - 1) & \textit{at } \Delta = \pm 2 \quad \textit{for odd branch order}. \end{cases}$$

Proof. Lemma 4.4 shows that all Δ (4.2) of periodic solutions of the sinh-Gordon equation of finite type fulfill (G)-(K). Due to Lemma 4.5 we can by suitable movements continuously deform an arbitrary Δ (4.2) obeying conditions (G)-(K) into the family of covering maps (4.6), which corresponds to spectral curves of geometric genus zero. In Lemma 4.3 it is shown that these deformations preserve those Δ, which correspond to periodic solutions of the sinh-Gordon equation of finite type. The form $d \ln \mu$ has $2g+2$ zeroes on the spectral curve. The formula for the genus is obtained by computing the order of the roots of $d \ln \mu$ on $\mu^2 - \mu\Delta(\kappa) + 1 = 0$. □

5. One-sided Alexandrov embeddings in $\mathbb{S}^3$

In this section we consider one-sided Alexandrov embeddings of general manifolds N and M. Some statements apply only to one-sided Alexandrov embeddings with constant mean curvature, but we do not use the special properties of CMC cylinders of finite type. We provide sufficient conditions which allow us to perturb surfaces which are one-sided Alexandrov embedded inside of a collar of the unperturbed surfaces into surfaces which remain one-sided

Alexandrov embedded. For this purpose we have to ensure that the surfaces have collars with depths uniformly bounded from below. Making use of the fact that there are no complete stable minimal surfaces in $\mathbb{S}^3$, we present in Lemma 5.8 a crucial technical result communicated to us by Harold Rosenberg [57]: If both principal curvatures of an Alexandrov embedded CMC surface are uniformly bounded, then the cut locus function is bounded from below by a positive number.

We consider one-sided Alexandrov embedded cylinders in $\mathbb{S}^3$. In the literature we only found the notion of Alexandrov embeddings for compact domains on the one hand, and the concept of properly Alexandrov embedded immersions from open manifolds into open Riemannian manifolds on the other hand. Since we are interested in immersions of open manifolds into the compact Riemannian manifold $\mathbb{S}^3$, we make the following

Definition 5.1. *A* **one-sided Alexandrov embedding** *in $\mathbb{S}^3$ is a smooth immersion f from a connected 3-manifold N with connected boundary $M = \partial N$ to $\mathbb{S}^3$ with the following properties:*

(i) *The mean curvature of M in $\mathbb{S}^3$ with respect to the inward normal is non-negative everywhere.*
(ii) *The manifold N is complete with respect to the metric induced by f.*

An immersion $f : N \to \mathbb{S}^3$ obeying condition (ii) is called an **Alexandrov embedding**.

A fixed orientation of $\mathbb{S}^3$ induces on N and $M = \partial N$ an orientation. Conversely, if M is endowed with an orientation, then there exists a unique normal, which points inward to the side of M in $\mathbb{S}^3$, which induces on the boundary M the given orientation of M. In this sense the orientation of M determines the inner normal of N.

For each point $p \in M$ of a hypersurface of a Riemannian manifold N there exists a unique arc length parameterized geodesic $\gamma(p, \cdot)$ emanating from $p = \gamma(p, 0)$ and going in the direction of the inward normal at p. Such geodesics are called **inward M-geodesics** [24].

Let $\gamma(p, \cdot)$ be an inward M-geodesic. Points $q \in N$ in the ambient manifold that are 'close to one side' of M can thus be uniquely parameterized by (p, t) where $p \in M$ and $q = \gamma(p, t)$ for some inward M-geodesic $\gamma(p, \cdot)$ and some $t \in \mathbb{R}_0^+$. The value of t is the geodesic distance of q to M. Extending the geodesic further into N it might eventually encounter a point past which $\gamma(p, t)$ no longer minimizes the distance to M. Such a point is called a cut point. The cut locus of M in N consists of the set of cut points along all inward M-geodesics. We define the **cut locus function** as the geodesic distance of the cut point to M:

$$c : M \to \mathbb{R}^+, \quad p \mapsto c(p), \qquad \text{such that } \gamma(p, c(p)) \text{ is the cut point.} \tag{5.1}$$

If we want to stress the dependence on f we decorate γ and c with index f. A known fact from Riemannian Geometry ([24, Lemma 2.1]) asserts that a cut point is either the first focal point on an inward M-geodesic, or is the intersection point of two shortest inward M-geodesics of equal length.

For a one-sided Alexandrov embedding $f : N \to \mathbb{S}^3$ of a 3-manifold N with boundary $\partial N = M$, the inward M-geodesics give us a parametrization of N, which we call **generalized cylinder coordinates**:

$$\gamma_f : \{(p, t) \in M \times \mathbb{R} \mid 0 \leq t \leq c_f(p)\} \to N\,. \tag{5.2}$$

The restriction of γ_f to $\{(p, t) \in M \times \mathbb{R} \mid 0 \leq t < c_f(p)\}$ is a diffeomorphism onto the complement of the cut locus. The cut locus is homeomorphic to the quotient space $M/\sim_f$ with

the following equivalence relation on M:

$$\begin{aligned} p \sim_f q \qquad &\Longleftrightarrow \qquad c_f(p) = c_f(q) \quad \text{and} \quad \gamma_f(p, c_f(p)) = \gamma_f(q, c_f(q)) \\ &\Longleftrightarrow \qquad \gamma_f(p, c_f(p)) = \gamma_f(q, c_f(q)). \end{aligned}$$

For all $p \in M$ we denote the corresponding equivalence classes by

$$[p]_f = \{q \in M \mid \gamma_f(p, c_f(p)) = \gamma_f(q, c_f(q))\}. \tag{5.3}$$

Lemma 5.2. *Let $f : N \to \mathbb{S}^3$ and $\tilde{f} : \tilde{N} \to \mathbb{S}^3$ be two one-sided Alexandrov embeddings. The orientation of $\mathbb{S}^3$ induces orientations on N, ∂N, $\tilde{N}$ and $\partial \tilde{N}$. If the two oriented boundaries $\partial N = M = \partial \tilde{N}$ and the two restrictions to the boundaries $f|_M = \tilde{f}|_M$ coincide, then there exists a diffeomorphism $\Psi : \tilde{N} \to N$, whose restriction to M is the identity map of M, such that $\tilde{f} = f \circ \Psi$.*

Proof. The immersions f and $\tilde{f}$ induce on N and $\tilde{N}$ two Riemannian metrics g and $\tilde{g}$, which coincide on M. The Riemannian metrics induce on N and $\tilde{N}$ two metrics d and $\tilde{d}$. Let c_f and $c_{\tilde{f}}$ denote the cut locus functions of the submanifold M in the two Riemannian manifolds (N, g) and $(\tilde{N}, \tilde{g})$. The generalized cylinder coordinates (5.2) define diffeomorphisms γ_f and $\gamma_{\tilde{f}}$ of

$$L = \{(p, t) \in M \times \mathbb{R} \mid 0 \leq t < \min\{c_f(p), c_{\tilde{f}}(p)\}\}$$

onto open subsets of N and $\tilde{N}$. First we claim that these diffeomorphisms γ_f and $\gamma_{\tilde{f}}$ together with the metrics d and $\tilde{d}$ induce on L the same metrics. In fact, the subset of $L \times L$ on which the metrics coincide is open and closed and therefore all of $L \times L$. Consequently both cut locus functions c_f and $c_{\tilde{f}}$ coincide, and for all points $p \in M$ the corresponding classes $[p]_f = [p]_{\tilde{f}}$ (5.3) coincide too. Hence the diffeomeorphism $\gamma_f \circ \gamma_{\tilde{f}}^{-1}$ from the complement of the cut locus in $\tilde{N}$ onto the complement of the cut locus in N extends to a homeomorphism Ψ from $\tilde{N}$ to N. By definition of the cylinder coordinates the immersions $f \circ \gamma_f$ and $\tilde{f} \circ \gamma_{\tilde{f}}$ from L into $\mathbb{S}^3$ coincide. Hence we have $\tilde{f} = f \circ \Psi$. Since f and $\tilde{f}$ are immersions, Ψ is a diffeomorphism form $\tilde{N}$ onto N. □

Definition 5.3. *We call an immersion $f : W \to \mathbb{S}^3$ of a connected 3-manifold W with connected boundary $V = \partial W$ a* **local one-sided Alexandrov embedding**, *if the following hold:*

(i) *The mean curvature of V in $\mathbb{S}^3$ with respect to the inward normal is non-negative everywhere.*
(ii) *All inward V-geodesics exist in W until they reach the cut locus* (5.1).
(iii) *The generalized cylinder coordinates γ_f* (5.2) *are surjective.*

An immersion obeying conditions (ii)-(iii) is called a **local Alexandrov embedding**.

If $f : N \to \mathbb{S}^3$ is an Alexandrov embedding, and $V \subset M = \partial N$ is an open subset, which contains for all $p \in V$ the classes $[p]_f$ (5.3), then the restriction $f|_W$ of f to

$$W = \{\gamma_f(p, t) \in N \mid p \in V \text{ and } 0 \leq t \leq c_f(p)\}$$

is a local Alexandrov embedding. The proof of Lemma 5.2 carries over to the following situation:

Corollary 5.4. *Let $f : W \to \mathbb{S}^3$ and $\tilde{f} : \tilde{W} \to \mathbb{S}^3$ be two local Alexandrov embeddings. If the oriented boundaries $\partial W = V = \partial \tilde{W}$ and $f|_V = \tilde{f}|_V$ coincide, then there exists a diffeomeorphism $\Psi : \tilde{W} \to W$, whose restriction to V is the identity map of V, such that $\tilde{f} = f \circ \Psi$.* □

We shall prove that 'one-sided Alexandrov embeddedness' is an open condition, which will allow us to study deformation families of one-sided Alexandrov embeddings. The main tool is a

general perturbation technique of Alexandrov embeddings, which we call **collar perturbation**. We consider perturbations $\tilde{f}$ of a given smooth immersion $f : M \to \mathbb{S}^3$, which are 'small' with respect to the C^1-topology on the space of immersions from M into $\mathbb{S}^3$. For this purpose we use the trivialization of the tangent bundle $T\mathbb{S}^3 \cong \mathrm{SU}_2 \times \mathfrak{su}_2$ by left invariant vector fields.

Remark 5.5. *For sake of simplicity we denote the derivative considered as a smooth function into $\mathfrak{su}_2$ by f' instead of $f^{-1}df$, and endow $\mathfrak{su}_2$ with the norm $\|X\| = (-\frac{1}{2}\mathrm{tr}(X^2))^{\frac{1}{2}}$.*

Lemma 5.6. *Let $f : M \to \mathbb{S}^3$ be an immersion inducing on the 2-manifold M a complete Riemannian metric. If the absolute values of both principal curvatures are bounded by $\kappa_{\max} > 0$, then for some $r > 0$ depending only on $\kappa_{\max}$ the exponential map $\exp_p$ is at all points $p \in M$ a diffeomorphism from $B(0, r) \subset T_pM$ onto an open neighbourhood of p. Furthermore, for all $\epsilon > 0$ there exists a $\delta > 0$ depending only on ϵ and $\kappa_{\max}$, such that*

$$\|(f \circ \exp_p)'(v) - (f \circ \exp_p)'(0)\| < \epsilon \quad \text{for all } p \in M \text{ and all } v \in B(0, \delta) \subset T_pM.$$

Proof. Choose two radii $r, R \in (0, \pi)$ such that the following two conditions hold:

(i) $r + R \leq \arctan(\kappa_{\max}^{-1})$.

(ii) Let $q^\pm$ be the centers of two spheres of radius R in $\mathbb{S}^3$ touching each other at the center p of an open ball $B(p, r) \subset \mathbb{S}^3$. All geodesics in $\mathbb{S}^3$ emanating from q^+ or q^- can intersect transversally the sphere of radius R around q^- and q^+ respectively, inside of $B(p, r)$.

For $R < \arctan(\kappa_{\max}^{-1})$ both conditions can be fulfilled for small r. For all $p \in M$ there are exactly two points $q^\pm$ on the M-geodesic through p, whose distance to p are equal to R. Both spheres of radius R and centers $q^\pm$ in $\mathbb{S}^3$ touch M at p. The absolute values of both principal curvatures are not larger than the principal curvatures of all spheres in $\mathbb{S}^3$ with radius smaller than $\arctan(\kappa_{\max}^{-1})$. Hence on all geodesics of M emanating from p the distances to both centers $q^\pm$ are monotone increasing, as long as these distances are smaller than $\arctan(\kappa_{\max}^{-1})$. Due to condition (i) this is the case for all points on the geodesic, whose geodesic distance to p is not larger than r. Hence the ball $B(p, r) \subset M$ is mapped by f into the complement of both balls $B(q^\pm, R) \subset \mathbb{S}^3$. We parameterize the points of $B(p, r) \subset M$ by the intersection points of the shortest geodesics connecting these points with $q^\pm$ with the spheres of radius R around $q^\pm$, respectively. Due to the second condition (ii) these parameters are smooth. Hence $\exp_p$ is on $B(0, r) \subset T_pM$ a diffeomorphism onto an open neighbourhood of p. Since on all M-geodesics through p the distances to both centers $q^\pm$ is monotone increasing, the uniform estimate $\|(f \circ \exp_p)'(v) - (f \circ \exp_p)'(0)\| < \epsilon$ holds for all $p \in M$ and all sufficiently small $v \in T_pM$. □

Corollary 5.7. *Let $f : N \to \mathbb{S}^3$ be a one-sided Alexandrov embedding with uniform upper bound $\kappa_{\max}$ on both principal curvatures. Then there exist constants $0 < c_{\max} < \arctan(\kappa_{\max}^{-1})$ and $L > 0$ depending only on $\kappa_{\max}$ such that for all $p \in M = \partial N$ with $c_f(p) < c_{\max}$ the set $[p]_f$ (5.3) contains exactly two points. Furthermore, the angle between both corresponding inward M-geodesics at the cut locus $\gamma_f(p, c_f(p))$ is larger than $\pi - L \cdot c_f(p)$.*

Proof. All 2-spheres of radius $t \in (0, \frac{\pi}{2}]$ in $\mathbb{S}^3$ have principal curvatures both equal to $\kappa = \cot(t)$ (for the computations see e.g. Montiel and Ros [50]). For every $p \in M$ there is a unique inward M-geodesic $\gamma(p, \cdot)$. Evaluating these geodesics at some $t \in \mathbb{R}_0^+$ gives a hypersurface

$$M_t = \bigcup_{p \in M} \gamma(p, t) \tag{5.4}$$

in N. This hypersurface is smooth as long as $\gamma(p, t)$ does not pass through a focal point. If κ_1 and κ_2 are the two principal curvatures of M in p with respect to the inner normal, then

the corresponding principal curvatures of M_t at $\gamma(p,t)$ are given by $\cot(\arctan(\kappa_1^{-1}) - t)$ and $\cot(\arctan(\kappa_2^{-1}) - t)$. The focal point on $\gamma(p,\cdot)$ is at the value of t given by

$$t_{\text{foc}} = \arctan\left((\max\{\kappa_1,\, \kappa_2\})^{-1}\right) \geq \arctan(\kappa_{\max}^{-1}). \tag{5.5}$$

At all $p \in M$ with $c_f(p) < \arctan(\kappa_{\max}^{-1})$ the cut locus $\gamma_f(p, c_f(p))$ cannot be a focal point. Hence $[p]_f$ has to contain at least one other element $q \in M \setminus \{p\}$. Choose two radii r, R as in the proof of Lemma 5.6. The two spheres in $\mathbb{S}^3$ of radius R, which touch M at p and q outside of N, must not intersect each other at distances $< r$ to p and q, respectively. Hence there exist $c_{\max} > 0$ and $L > 0$, such that for $c_f(p) < c_{\max}$ the angle between the inward M-geodesics $\gamma_f(p,\cdot)$ and $\gamma_f(q,\cdot)$ at the cut locus is larger than $\pi - L \cdot c_f(p)$. If $c_{\max}$ is small enough, this implies that any element of $[p]_f$ belongs to one of the balls $B(p,r)$ or $B(q,r)$ in M. If $c_f(p)$ is smaller than R, then due to Lemma 5.6 $[p]_f \cap B(p,r) = \{p\}$ and $[p]_f \cap B(q,r) = \{q\}$. □

We thank Harold Rosenberg for communicating the following result to us. This arose through discussions with Antonio Ros and Harold Rosenberg, and makes use of a technique for constructing a global non-negative, non-trivial Jacobi field that was recently also employed by Meeks, Perez and Ros [48].

Lemma 5.8 (Rosenberg's Lemma)**.** *Let $f : N \to \mathbb{S}^3$ be a one-sided Alexandrov embedding with constant mean curvature, and principal curvatures bounded by $\kappa_{\max} > 0$. Then the cut locus function is bounded from below by* $\arctan(\kappa_{\max}^{-1})$.

Proof. The mean curvature of the hypersurface (5.4)

$$H(t) = \tfrac{1}{2}\left(\cot\left(\arctan(\kappa_1^{-1}) - t\right) + \cot\left(\arctan(\kappa_2^{-1}) - t\right)\right) \tag{5.6}$$

is positive for all $t \in (0,\, t_{\text{foc}})$, and strictly monotone increasing, since

$$H'(t) = \tfrac{1}{2}\left(\sin^{-2}(\arctan(\kappa_1^{-1}) - t) + \sin^{-2}(\arctan(\kappa_2^{-1}) - t)\right) > 0\,.$$

Let c_f denote the cut locus function (5.1). If there exists a point $p \in M$ for which $c_f(p) < \arctan(\kappa_{\max}^{-1}) \leq t_{\text{foc}}$, then two inward M-geodesics $\gamma(p,\cdot)$, $\gamma(q,\cdot)$ through p, $q \in M$ respectively, have to intersect at a distance of $c_f(p)$ from M, and thus $c_f(p) = c_f(q)$. Hence, if there exists a point $p \in M$ with $c_f(p) < \arctan(\kappa_{\max}^{-1})$ then M_t intersects itself for a value of $t < \arctan(\kappa_{\max}^{-1})$ over two points p, $q \in M$. Let

$$c_0 = \inf\{t \mid M_t \text{ intersects over two points of } M\}.$$

Since over all points $p \in M$ the mean curvature of M_t is positive for all $0 < t < \arctan(\kappa_{\max}^{-1})$ with respect to the inner normals, the surfaces M_t cannot touch each other from different sides over two points for $0 < t < \arctan(\kappa_{\max}^{-1})$.

A normal graph in $\mathbb{S}^3$ over a domain in a geodesic sphere $\mathbb{S}^2$ is the graph of the composition of the exponential map with a section of the normal bundle on this domain, see Fornari, deLira and Ripoll [21]. By Lemma 5.6 there exists a $r > 0$ depending only on $\kappa_{\max}$ such that all $p \in M$ have open neighbourhoods in M, which are normal CMC graphs over the ball $B(p,\, r)$ inside the unique geodesic 2-sphere $\mathbb{S}^2$, which touches M at p. Due to Arzelà-Ascoli, and the a priori gradient bound from Proposition 4.1 in [21], every bounded sequence of normal CMC graphs over $B(p,\, r) \subset \mathbb{S}^2$ has a convergent subsequence. Now let $(p_k)_{k\in\mathbb{N}}$ be a sequence in M with

$$\lim_{k\to\infty} c_f(p_k) = c_0 = \inf\left\{c_f(p) \mid p \in M\right\}\,.$$

Then there exists a sequence Θ_k of isometries of $\mathbb{S}^3$ which transform each point p_k into a fixed reference point $p_0 \in \mathbb{S}^3$, and the tangent plane of M at p_k into the tangent plane of a fixed geodesic sphere $\mathbb{S}^2_{p_0} \subset \mathbb{S}^3$ which contains p_0. This sequence of isometries transforms neighbourhoods U_k of $p_k \in M$ into normal CMC graphs $\Theta_k[U_k]$ over $B(p_0,\, r)$. By passing

to a subsequence we may achieve that these graphs converge to a normal CMC graph U over $B(p_0, r) \subset \mathbb{S}^2_{p_0}$, which is tangent to $\mathbb{S}^2_{p_0}$ at p_0. Due to Corollary 5.7, the sets $[p_k]_f$ contain besides p_k another point q_k for large k. Furthermore, the sequence of isometries Θ_k transforms the sequence of geodesic 2-spheres tangent to M at q_k into a converging sequence of spheres with limit $\mathbb{S}^2_{q_0}$. This sphere contains the limit $q_0 = \lim \Theta_k(q_k)$ with distance $\operatorname{dist}(p_0, q_0) = 2c_0$. The shortest geodesic connecting p_0 and q_0 intersects orthogonally both geodesics spheres $\mathbb{S}^2_{p_0}$ and $\mathbb{S}^2_{q_0}$. By Corollary 5.7, for large k the points q_k have neighbourhoods V_k, whose transforms $\Theta_k[V_k]$ are normal CMC graphs over $B(q_0, r) \subset \mathbb{S}^2_{q_0}$. By passing again to a subsequence the normal CMC graphs $\Theta_k[V_k]$ converge to a normal CMC graph V tangent to $Sp^2_{q_0}$ at q_0.

The transformed inward M-geodesics nearby p_k and q_k converge to normal geodesics of these two limiting CMC surfaces U and V in $\mathbb{S}^3$. If we shift both limiting CMC surfaces by c_0 along their normal geodesics as in (5.4) we obtain two surfaces touching each other from different sides at the limit of the transformed cut points $\lim \Theta_k(\gamma_f(p_k, c(p_k))) = \lim \Theta_k(\gamma_f(q_k, c(q_k)))$. Hence the shifted surfaces cannot have positive mean curvature with respect to the inner normal. This implies $c_0 = 0$ and $H = 0$. In this case $\mathbb{S}^2_{p_0} = \mathbb{S}^2_{q_0}$ and the two minimal surfaces U and V are tangent at $p_0 = q_0$ and are graphs over the same domain $B(p_0, r)$. By Hopf's maximum principle both limiting minimal normal graphs U and V coincide.

In this case, due to Corollary 5.7, for large k neighbourhoods $W_k \subset M$ of q_k are normal graphs over $U_k \subset M$. Let n_k be the corresponding sequence of non-negative sections of the inward normal bundle of the sequence $\Theta_k[U_k]$ of minimal graphs over $B(p_0, r)$. Then $\Theta_k[W_k]$ are normal graphs over the sequence $\Theta_k[U_k]$ of normal graphs over $B(p_0, r)$. Then

$$\lim_{k\to\infty} \frac{n_k}{\sup\{|n_k(q)| \mid q \in \Theta_k[U_k]\}}$$

converges to a non-trivial non-negative Jacobi field on the limiting minimal graph U over $B(p_0, r)$. This argument is a slight variation of an observation by Meeks, Perez and Ros [48].

Since the limits of both sequences of normal minimal graphs coincide, the function $|n_k|$ converges on $B(p_0, r)$ uniformly to zero. Hence we may repeat the same line of argument at all points of the boundary of these minimal normal graphs $\Theta_k[U_k]$ over $B(p_0, r)$. By passing to a diagonal subsequence like in the Arzelà-Ascoli theorem we may extend U to a complete minimal surface in $\mathbb{S}^3$ with a non-negative Jacobi field which does not vanish identically. (Note that due to unique continuation of Jacobi fields [73] the limit of the re-scaled difference of both normal graphs stays bounded on all connected compact subsets of the complete minimal surface, whenever it is bounded on one compact subset.) A non-trivial non-negative Jacobi field implies that the spectrum of the Jacobi operator is non-negative (Allegretto-Piepenbrink Theorem, see Fischer-Colbrie and Schoen [20] Theorem 1 and Davies [15] Lemma 4.1). Therefore the limiting complete minimal surface is stable. But by a result of Fischer-Colbrie [20] (see also Corollary 3 in Schoen [61]) there exist no complete stable minimal surfaces in $\mathbb{S}^3$. This contradicts $c_0 < \arctan(\kappa_{\max}^{-1})$. $\square$

Later we apply the collar deformation to bounded open subsets $W \subset N$ of one-sided Alexandrov embeddings $f : N \to \mathbb{S}^3$. Hence we shall find bounded open subsets $V \subset M = \partial N$, which contain for all $p \in V$ the classes $[p]_f$ (5.3). For this purpose we need a **chord-arc** bound:

Lemma 5.9. *Let $f : N \to \mathbb{S}^3$ be a one-sided Alexandrov embedding with second fundamental form $\mathfrak{h}$ with respect to the inner normal $\mathfrak{N}$. If c is a lower bound on the cut locus functions c_f* (5.1) *and D a uniform bound on the covariant derivative of the second fundamental form:*

$$|(\nabla_X \mathfrak{h})(X, X)| = |\nabla_X(\mathfrak{h}(X, X)) - 2\mathfrak{h}(\nabla_X X, X)| \leq D|X|^3 \quad \text{for all } X \in TM, \tag{5.7}$$

then there exists $C > 0$ depending only on c and D, such that

$$\mathrm{dist}_N(p,q) \leq \mathrm{dist}_M(p,q) \leq C\,\mathrm{dist}_N(p,q) \quad \textit{for all } p,q \in M.$$

Proof. For all $p,q \in M$ we have $\mathrm{dist}_N(p,q) \leq \mathrm{dist}_M(p,q)$. In general, these distances do not coincide. We shall construct a path from p to q of length at most $C\,\mathrm{dist}_N(p,q)$. Due to (Rinow [55], pages 172 and 141) the points p and q are joined by a shortest path in N. In case this shortest path touches at some points the boundary [3, Theorem 1.] we decompose it into pieces. The boundary points of a shortest path might have accumulation points. But any point of a shortest path, which is not a boundary point, belongs to a unique geodesic piece in N, which has only two boundary points at both ends. Hence it suffices to construct such a path for two points p and q, which are connected by a geodesic in N with only two boundary points at both ends.

Due to [24, Lemma 2.1] the cut locus function $c_f(p)$ (5.1) is for all $p \in M$ not larger than the first focal point t_{foc} (5.5). Hence both principal curvatures are uniformly bounded by $\kappa_{\max} = \cot(c)$. If $\mathrm{dist}_N(p,q) \leq c$, then

$$\mathrm{dist}_N(\gamma_f(p, \mathrm{dist}_N(p,q)), q) \geq \mathrm{dist}_N(p,q) \quad \text{and} \quad \mathrm{dist}_N(\gamma_f(q, \mathrm{dist}_N(p,q)), p) \geq \mathrm{dist}_N(p,q).$$

Hence the angles between the geodesic connecting p and q, and the inward M-geodesics at p and q are not smaller than the angles of the triangle in $\mathbb{S}^2$ with three sides of length $\mathrm{dist}_N(p,q)$, which is larger than $\frac{\pi}{3}$. If we reduce the length of $\mathrm{dist}_N(p,q)$ along the gradient flow of dist_N on $M \times M$, then we obtain a path in $M \times M$ from (p,q) to the diagonal in $M \times M$ of length smaller than $2\,\mathrm{dist}_N(p,q)$. This shows the claim for short distances $\mathrm{dist}_N(p,q) \leq c$. On the other hand, all geodesics γ in N starting at p, which do not meet M for distances $d \in (0,\pi)$ meet each other at the antipode of p. The exponential map of N maps a unique half space of T_pN into N. If the pre-image of $B(p,\pi) \subset N$ with respect to $\exp_p$ contains this half space, then due to Hopf's maximum principle (see e.g. [19]) $B(p,\pi) \subset M$ is a geodesic sphere in $\mathbb{S}^3$ and the statement is obvious. Otherwise there exists a geodesic starting at p, which touches M for some $t \in (0,\pi)$. In particular, if $\mathrm{dist}_N(p,q) \geq \pi$, then there exists $\tilde{q} \in M$ with

$$\mathrm{dist}_N(p,\tilde{q}) < \pi \quad \text{and} \quad \mathrm{dist}_N(p,q) \geq \mathrm{dist}_N(p,\tilde{q}) + \mathrm{dist}_N(\tilde{q},q).$$

Hence we may assume $c < \mathrm{dist}_N(p,q) < \pi$. Let $\chi_p,\, \chi_q \in [0, \frac{\pi}{2}]$ denote the angles in T_pN and T_qN between the inward geodesic γ connecting p and q and the inward normal to M, respectively. In this proof we shall consider smooth families of geodesics γ connecting two smooth paths $s \mapsto p(s)$ and $s \mapsto q(s)$ in M parameterized by a real parameter s. For fixed s the geodesic is parameterized by the real parameter t. The derivatives with respect to s are denoted by prime and the derivatives with respect to t by dot. Let $|p'|$ and $|q'|$ denote the lengths of the tangent vectors p' and q' with respect to the Riemannian metric. The function

$$|\nabla\,\mathrm{dist}_N(p,q)| = \sup\left\{\tfrac{|\,\mathrm{dist}'_N(p,q)|}{|p'|+|q'|} \mid p' \in T_pM, q' \in T_qM\right\} = \max\{|\tan(\chi_p)|, |\tan(\chi_q)|\} \tag{5.8}$$

is the length of the gradient of the function dist_N on $M \times M$. The geodesic γ extends in $\mathbb{S}^3$ to a closed geodesic. For any tangent vector $(p',q') \in T_pM \times T_qM$ there exists a Killing field ϑ on $\mathbb{S}^3$, which moves the closed geodesic γ in such a way, that the intersection points at p and q moves along p' and q', respectively. Conversely, all Killing fields ϑ generate a one-dimensional group of isometries of $\mathbb{S}^3$. Let $s \mapsto \gamma_\vartheta(s,\cdot)$ denote the corresponding family of geodesics and $s \mapsto (p_\vartheta(s), q_\vartheta(s))$ the corresponding intersection points with M. In order to proceed we need

Lemma 5.10. *For all $\kappa_{\max} > 0$ and $0 < c < \pi$ there exist $\epsilon, \delta > 0$ and $0 < s_0 < \frac{3}{2}$ with the following property: For all $(p,q) \in M \times M$ with $c \leq \mathrm{dist}_N(p,q) < \pi$ and $\max\{\tan(\chi_p), \tan(\chi_q)\} \leq \epsilon$ there exists a non trivial Killing field ϑ, such that $d : s \mapsto d(s) = \mathrm{dist}_N(p_\vartheta(s), q_\vartheta(s))$ obeys*

$$d'(s) \leq 0 \quad d''(s) \leq -\delta \cos\left(\tfrac{d(s)}{2}\right) \quad |p'_\vartheta(s)| + |q'_\vartheta(s)| \leq 3\cos\left(\tfrac{d(s)}{2}\right) \quad \textit{for all } s \in [0, s_0]. \tag{5.9}$$

Proof. We shall construct a Killing field ϑ with the desired properties, which rotates γ around two antipodes of γ. The corresponding rotated geodesics $\gamma_\vartheta(s,\cdot)$ belong to a unique geodesic 2-sphere $\mathbb{S}^2 \subset \mathbb{S}^3$. The corresponding paths $s \mapsto p(s)$ and $s \mapsto q(s)$ move along the intersection of this sphere $\mathbb{S}^2$ with M. Hence we can calculate all derivatives on this sphere.

We parameterize this sphere by the real parameter s of the family $s \mapsto \gamma_\vartheta(s,\cdot)$ of rotated geodesics, and the real parameter t of these geodesics. We choose the equator as the start points $\gamma_\vartheta(s,0)$ with distance $\frac{\pi}{2}$ to the rotation axis. The vector fields ϑ and the geodesic vector field $\dot\gamma$ along the geodesics $\gamma_\vartheta(s,\cdot)$ form an orthogonal base of the tangent space $T\mathbb{S}^2$ of this sphere away from the zeroes of ϑ. The vector fields ϑ and $\dot\gamma$ have at (s,t) the scalar products

$$g(\vartheta,\vartheta) = \cos^2(t) \qquad\qquad g(\vartheta,\dot\gamma) = 0 \qquad\qquad g(\dot\gamma,\dot\gamma) = 1.$$

Since $\dot\gamma$ is a geodesic vector field the derivative $\nabla_{\dot\gamma}\dot\gamma$ vanishes. Moreover, the mean curvature of the integral curve of ϑ starting at (s,t) is equal to $\tan(t)$. Therefore at (s,t) we have

$$\begin{aligned} &\nabla_\vartheta\vartheta = \cos^2(t)\tan(t)\dot\gamma = \cos(t)\sin(t)\dot\gamma\,, && \nabla_\vartheta\dot\gamma = -\tan(t)\vartheta\,,\\ &\nabla_{\dot\gamma}\vartheta = -\tan(t)\vartheta\,, && \nabla_{\dot\gamma}\dot\gamma = 0\,. \end{aligned}$$

We parameterize a neighbourhood of the geodesic from p to q in such a way that the corresponding vector field $\dot\gamma$ points inward to N at p and outward of N at q, respectively. Let (s_p,t_p) and (s_q,t_q) be the coordinates of p and q, respectively. The Killing field ϑ induces along the paths $s \mapsto p(s)$ and $s \mapsto q(s)$ the vector fields

$$p' = \vartheta(p) - \dot\gamma(p)\frac{g(\mathfrak{N}(p),\vartheta)(p)}{g(\mathfrak{N}(p),\dot\gamma(p))} \qquad \text{and} \qquad q' = \vartheta(q) - \dot\gamma(q)\frac{g(\mathfrak{N}(q),\vartheta(q))}{g(\mathfrak{N}(q),\dot\gamma(q))}$$

with lengths $|p'| = \frac{|\cos(t_p)|}{\cos(\chi_p)}$ and $|q'| = \frac{|\cos(t_q)|}{\cos(\chi_q)}$. The derivative d' is equal to

$$d' = \frac{g(\mathfrak{N}(p),\,\vartheta(p))}{g(\mathfrak{N}(p),\,\dot\gamma(p))} - \frac{g(\mathfrak{N}(q),\,\vartheta(q))}{g(\mathfrak{N}(p),\,\dot\gamma(q))} = \frac{g(\mathfrak{N}(p),\,\vartheta(p))}{\cos(\chi_p)} - \frac{g(\mathfrak{N}(q),\,\vartheta(q))}{\cos(\chi_q)}.$$

Along the paths p and q with $X = p'$ and $X = q'$, respectively, we have at (s,t)

$$\begin{aligned} \nabla_X\frac{g(\mathfrak{N},\,\vartheta)}{g(\mathfrak{N},\,\dot\gamma)} &= \frac{g(\nabla_X\mathfrak{N},\,\vartheta) + g(\mathfrak{N},\,\nabla_X\vartheta)}{g(\mathfrak{N},\,\dot\gamma)} - \frac{g(\mathfrak{N},\vartheta)(g(\nabla_X\mathfrak{N},\,\dot\gamma) + g(\mathfrak{N},\,\nabla_X\dot\gamma))}{(g(\mathfrak{N},\,\dot\gamma))^2}\\ &= \frac{g(\nabla_X\mathfrak{N},X) + g(\mathfrak{N},\nabla_X\vartheta)}{g(\mathfrak{N},\dot\gamma)} - \frac{g(\mathfrak{N},\theta)g(\mathfrak{N},\nabla_X\dot\gamma)}{g(\mathfrak{N},\dot\gamma)^2}\\ &= -\frac{\mathfrak{h}(X,\,X)}{g(\mathfrak{N},\dot\gamma)} + \cos(t)\sin(t) + 2\tan(t)\left(\frac{g(\mathfrak{N},\,\vartheta)}{g(\mathfrak{N},\,\dot\gamma)}\right)^2. \end{aligned}$$

Hence the second derivative is equal to

$$\begin{aligned} d'' = &-\frac{\mathfrak{h}(p',\,p')}{\cos(\chi_p)} - \frac{\mathfrak{h}(q',\,q')}{\cos(\chi_q)} + \frac{\sin(2t_p) - \sin(2t_q)}{2}\\ &+ 2\tan(t_p)\left(\frac{g(\mathfrak{N}(p),\,\vartheta(p))}{g(\mathfrak{N}(p),\,\dot\gamma(p))}\right)^2 - 2\tan(t_q)\left(\frac{g(\mathfrak{N}(q),\,\vartheta(q))}{g(\mathfrak{N}(q),\,\dot\gamma(q))}\right)^2. \end{aligned}$$

If along the rotation of the geodesic for $s \in [0,s_0]$ the following inequalities are satisfied

$$(5.10) \qquad -\tfrac{\pi}{2} \le t_p \le 0, \quad 0 \le t_q \le \tfrac{\pi}{2}, \quad \tfrac{c}{2} \le t_q - t_p, \quad \text{and} \quad \max\{\tan(\chi_p),\tan(\chi_q)\} \le \tfrac{1}{\sqrt{3}},$$

then $\frac{c}{2} \leq d = t_q - t_p$, $\min\{\cos(\chi_p), \cos(\chi_q)\} \geq \frac{\sqrt{3}}{2}$ and the last two terms of d'' are bounded by

$$\left|\tan(t_p)\left(\frac{g(\mathfrak{N}(p),\, \vartheta(p))}{g(\mathfrak{N}(p),\, \dot\gamma(p))}\right)^2\right| \leq \sin(|t_p|)\cos(t_p)\tan^2(\chi_q) \leq \frac{\sin(2|t_p|)}{2\cdot 3}$$
$$\left|\tan(t_q)\left(\frac{g(\mathfrak{N}(q),\, \vartheta(q))}{g(\mathfrak{N}(q),\, \dot\gamma(q))}\right)^2\right| \leq \sin(|t_p|)\cos(t_p)\tan^2(\chi_q) \leq \frac{\sin(2|t_q|)}{2\cdot 3}.$$

Due to $\sin(2t_q)-\sin(2t_p) = 2\sin(t_q-t_p)\cos(t_p+t_q)$ and $\cos(t_p)+\cos(t_q) = 2\cos(\frac{t_q-t_p}{2})\cos(\frac{t_p+t_q}{2})$ we arrive at

$$d''(s) \leq -\frac{\mathfrak{h}(p',\, p')}{\cos(\chi_p)} - \frac{\mathfrak{h}(q',\, q')}{\cos(\chi_q)} - \sin(d)\cos(t_p+t_q)\left(1-\tfrac{2}{3}\right),$$
$$|p'|+|q'| = \frac{\cos(t_p)}{\cos(\chi_p)} + \frac{\cos(t_p)}{\cos(\chi_p)} \leq \tfrac{4}{\sqrt{3}}\cos\left(\tfrac{d}{2}\right)\cos\left(\tfrac{t_p+t_q}{2}\right) \leq 3\cos(\tfrac{d}{2}).$$

The assumption (5.10) implies $t_p \leq -d+\frac{\pi}{2}$ and $d-\frac{\pi}{2} \leq t_q$. For $d \in [\frac{\pi}{2}, \pi)$ we get $\cos(t_p) \leq \sin(d)$ and $\cos(t_q) \leq \sin(d)$. For $d \in [\frac{c}{2}, \frac{\pi}{2})$ we use $\cos(t_p) \leq 1$ and $\cos(t_q) \leq 1$ and obtain

$$\sin\left(\tfrac{c}{4}\right) \leq \tfrac{1}{2}\sin\left(\tfrac{c}{2}\right) \leq \tfrac{3\sqrt{3}}{8}\sin\left(\tfrac{c}{2}\right) \leq \sin(d)\min\left\{\tfrac{\cos(\chi_p)}{|p'|^2}, \tfrac{\cos(\chi_q)}{|q'|^2}\right\}.$$

The third inequality of (5.9) implies $2\sin(\frac{c}{4})\cos(\frac{d}{2}) \leq \sin(d)$. To sum up, the second and the third inequalities of (5.9) are implied by (5.10) and

$$\delta \leq \frac{1}{3}\sin\left(\tfrac{c}{4}\right)\cos(t_p+t_q), \qquad -\frac{\mathfrak{h}(p',\, p')}{|p'|^2} \leq \tfrac{1}{2}\delta \quad \text{and} \quad -\frac{\mathfrak{h}(q',\, q')}{|q'|^2} \leq \tfrac{1}{2}\delta. \tag{5.11}$$

We shall show first that there exists a vector field ϑ obeying at $s=0$

$$\delta \leq \frac{1}{6}\sin\left(\tfrac{c}{4}\right)\cos(t_p+t_q), \qquad -\frac{\mathfrak{h}(p',\, p')}{|p'|^2} \leq \tfrac{1}{4}\delta \quad \text{and} \quad -\frac{\mathfrak{h}(q',\, q')}{|q'|^2} \leq \tfrac{1}{4}\delta.$$

The Killing field ϑ is uniquely determined by two choices:

Firstly the choice of a geodesic sphere $\mathbb{S}^2 \subset \mathbb{S}^3$, which contains the closed geodesic from p to q.

Secondly a choice of the zeroes of ϑ, or equivalently a choice of the coordinates t_p and t_q with $t_q - t_p = d \mod \pi$. We start with $t_q = -t_p = \frac{d}{2}$.

We first choose a 2-sphere $\mathbb{S}^2 \subset \mathbb{S}^3$. This 2-sphere is uniquely determined by a choice of a line in $T_p\mathbb{S}^2 \cap T_pM$ which includes p', or equivalently a choice of a line in $T_q\mathbb{S}^2 \cap T_qM$ which includes q'. Since f is a one-sided Alexandrov embedding and both principal curvatures are uniformly bounded by $\kappa_{\max}$, the cone angles of those double cones in T_pM and T_qM, on which $\mathfrak{h}(X,X) \geq -\frac{1}{4}\delta|X|^2$, are not smaller than $\frac{\pi}{2}+\mathrm{O}(\delta)$. For sufficiently small $\epsilon \geq \max\{\tan(\chi_p)\tan(,\chi_q)\}$ these double cones correspond in the plane orthogonal to $\dot\gamma(p)$ in T_pN and in the plane orthogonal to $\dot\gamma(q)$ in T_qN to double cones with cone angles not smaller than $\frac{\pi}{2}$. Hence the intersection of both double cones is non empty.

In a second step we shall show that the inequalities (5.10) and (5.11) are satisfied for $s \in [0,\, s_0]$ with some $s_0 > 0$. Due to Lemma 5.6 and the assumption $\max\{\tan(\chi_p), \tan(\chi_q)\} \leq \epsilon$ there exists s_0 such that t_p and t_q do not reach the roots of ϑ for $s \in [0,\, s_0]$. Since the derivatives of $\cos(\chi_p)$, $\cos(\chi_q)$, t_p and t_q with respect to s are uniformly bounded, there exists $s_0 > 0$ such that the inequalities (5.10) and the first inequality of (5.11) are satisfied for $s \in [0,\, s_0]$. Due to (5.7) also the derivatives of $\mathfrak{h}(p',p')$ and $\mathfrak{h}(q',q')$ are uniformly bounded. Hence there exists $s_0 > 0$ only depending on c and $\kappa_{\max}$, such that the second and the third inequality of (5.9) are satisfied for $s \in [0,\, s_0]$.

Finally we have to satisfy the first inequality of (5.9). At the start point $s = 0$ this is always the case for one choice of the sign of ϑ. Now the second inequality of (5.9) implies the first. $\square$

Continuation of the proof of Lemma 5.9. By Lemma 5.10 there exists $\epsilon, \delta > 0$ and $0 < s_0 < \frac{2}{3}$, such that for all all $p, q \in M$ with $c \leq \operatorname{dist}_N(p,q) < \pi$ and $|\nabla \operatorname{dist}_N(p,q)| \leq \epsilon$ there exists a Killing field ϑ, along which the length d is reduced for $0 \leq s \leq s_0$. The inequality $-d'(s) \leq |p'(s)| + |q'(s)| \leq 3\cos(\frac{d(s)}{2}) \leq 3\cos(\frac{d(s_0)}{2})$ implies $\cos(\frac{d(0)}{2}) \geq (1 - \frac{3}{2}s_0)\cos(\frac{d(s_0)}{2})$. Hence we get

$$\frac{d(0) - d(s_0)}{\operatorname{dist}_M(p(s_0), p(0)) + \operatorname{dist}_M(q(s_0), q(0))} \geq \frac{\delta \frac{s_0^2}{2} \cos\left(\frac{d(0)}{2}\right)}{3s_0 \cos\left(\frac{d(s_0)}{2}\right)} \geq \delta \frac{s_0}{6} \left(1 - \tfrac{3}{2}s_0\right).$$

We apply this procedure again and again, until either $\operatorname{dist}_N(p,q) < c$ or $|\nabla \operatorname{dist}_N(p,q)| > \epsilon$. As long as the gradient (5.8) $|\nabla \operatorname{dist}_N(p,q)| > \epsilon$, the same estimate holds for the corresponding gradient on the Riemannian manifold $M \times M$. Furthermore, the corresponding gradient flow reduces dist_N with monotonic decreasing $\frac{\epsilon}{\sqrt{2}} \operatorname{dist}_M(p(s), p(0)) + \frac{\epsilon}{\sqrt{2}} \operatorname{dist}_M(q(s), q(0)) + \operatorname{dist}_N(p(s), q(s))$. In summary, all points $p, q \in M$ obey $\operatorname{dist}_M(p,q) \leq \max\{2, \frac{\sqrt{2}}{\epsilon}, \frac{12}{\delta(2s_0 - 3s_0^2)}\} \operatorname{dist}_N(p,q)$. $\square$

Lemma 5.11. *(Collar perturbation) For given $\kappa_{\max} > 0$ and $0 \leq H_{\max} \leq \arctan(\kappa_{\max}^{-1})$ there exist $\epsilon > 0$ and $R > 0$ with the following property: If $f : N \to \mathbb{S}^3$ is a one-sided Alexandrov embedding with constant mean curvature $0 \leq H \leq H_{\max}$ and principal curvatures bounded by $\kappa_{\max} > 0$, $p \in M$ is some point, and $\tilde{f} : M \to \mathbb{S}^3$ is an immersion with constant mean curvature $\tilde{H} \geq 0$ obeying*

$$\text{(5.12)} \quad \operatorname{dist}(f(q), \tilde{f}(q)) < \epsilon \text{ and } \|f'(q) - \tilde{f}'(q)\| < \epsilon \quad \text{for all } q \in B(p,R) \text{ and } |H - \tilde{H}| < \epsilon.$$

Then $\tilde{f}$ extends to a local one-sided Alexandrov embedding $\tilde{f} : W \to \mathbb{S}^3$, whose boundary $V = \partial W \subset B(p,R)$ is an open neighbourhood of p.

Proof. Due to Rosenberg's Lemma 5.8 $c = \arctan(\kappa_{\max}^{-1})$ is a lower bound of the cut locus function (5.1). The generalized cylinder coordinates (5.2) define a diffeomorphism γ_f of $M \times [0, c)$ onto an open subset of N, which is a collar. Any lower bound on the cut locus function (5.1) is also a lower bound on the distance to the first focal point (5.5) on the inward M-geodesics. Since f is a one-sided Alexandrov embedding, the absolute values of the negative principal curvatures are smaller than the positive principal curvatures. Consequently, due to the formula (5.6), the distances to the first focal points on the outward M-geodesics are not smaller than the distances to the first focal points on the inward M-geodesics. Hence the normal variation defines an immersion of $(-c, c) \times M$ into $\mathbb{S}^3$. In particular, the induced metric makes $(-c, c) \times M$ into a Riemannian manifold with constant sectional curvature equal to one. For all elements of this manifold the cylinder coordinates, i.e. the distances to $M \simeq \{0\} \times M$ and the nearest point in M, are uniquely defined. Hence we can glue $(-c, c) \times M$ along $\gamma_f([0, c) \times M)$ to N, and obtain a larger 3-manifold $\hat{N} \supset N$ without boundary, such that the generalized cylinder coordinates (5.2) extend to a diffeomorphism $\hat{\gamma}_f : (-c, c) \times M \to \hat{N}$ and the immersion $f : N \to \mathbb{S}^3$ extends to an immersion $\hat{f} : \hat{N} \to \mathbb{S}^3$.

The balls with radius c around all elements of $M \subset \hat{N}$ are isometric to open balls in $\mathbb{S}^3$ of radius c. For $\epsilon < c$ all immersions $\tilde{f}$ obeying the first inequality of (5.12) can be represented uniquely as the composition of a smooth section $B(p,R) \to T\hat{N}|_{B(p,R)}$ with $\hat{f} \circ \exp_{\hat{N}}$. Here $\exp_{\hat{N}}$ denotes the exponential map of $\hat{N}$. Hence for all immersions $\tilde{f}$ obeying the first inequality of (5.12) there exists a smooth embedding $i : B(p,R) \hookrightarrow \hat{N}$, such that $\tilde{f}$ is equal to $\hat{f} \circ i$. Due to (5.12)

Φ induces on $B(p,R)$ a Riemannian metric denoted by dist_O nearby the original metric. Hence there exists a constant C_1 depending only on ϵ such that

$$\mathrm{dist}_O(i(q),i(q')) \le C_1 \,\mathrm{dist}_M(q,q') \quad \text{for all } q,q' \in B(p,R).$$

We denote the image of this embedding i by O as an oriented submanifold of $\hat N$. We shall identify $\tilde f|_{B(p,R)}$ with the immersion $\hat f|_O$. For all $q \in O$ let $t \mapsto \gamma_{\tilde f}(q,t)$ denote the inward O-geodesics in $\hat N$. Since O is not complete we have to be careful with the cut locus.

Due to Lemma 5.6 and Theorem 1.2 in [21] there exists $r > 0$ depending only on $\kappa_{\max}$ and $H_{\max}$, such that all $q \in M$ belong to open domains in M, which are normal CMC graphs over the ball $B(q,r)$ in the unique geodesic sphere $\mathbb{S}^2 \subset \mathbb{S}^3$, which touches M at q. Furthermore, for sufficiently small ϵ the same is true for all $q \in \overline{B(p,R-r)}$ and the perturbed immersion $\tilde f|_{B(p,R)}$. We conclude that at all $q \in M$ the second fundamental form $\mathfrak{h}$ of f obeys (5.7) with D depending only on $\kappa_{\max}$ and $H_{\max}$. Moreover, for sufficiently small ϵ the principal curvatures of the perturbation $\tilde f$ are bounded at all $q \in \overline{B(p,R-r)}$ by a constant depending only on $\kappa_{\max}$ and $H_{\max}$. Finally, again at all $q \in \overline{B(p,R-r)}$ the second fundamental form $\tilde{\mathfrak{h}}$ of $\tilde f$ obeys (5.7) with a constant D depending only on $\kappa_{\max}$ and $H_{\max}$. Due to Lemma 5.9 there exists a corresponding bound on the chord-arc ratio $C_2 > 0$ depending only on $\kappa_{\max}$ and $H_{\max}$.

All cut locus functions of one-sided Alexandrov embeddings are uniformly bounded from above by $\frac{\pi}{2}$, since otherwise a sphere with negative principal curvatures touches M inside of N contradicting Hopf's maximum principle. Now we choose $R = C_1(C_2 2\pi + \epsilon + r)$ and denote $U = B(i(p), C_2\pi + \epsilon) \subset O$. For all $q \in U \subset O$, we have

$$\left\{q' \in O \mid \exists t \in [0, \tfrac{\pi}{2}] \text{ with } \mathrm{dist}_{\hat N}(\gamma_{\tilde f}(q,\,t),\,q') \le t\right\} \subset \left\{q' \in O \mid \mathrm{dist}_{\hat N}(q,\,q') \le \pi\right\} \subset B(q,\,C_2\pi)\,.$$

Therefore, for all $q \in U$ the cut locus function $c_{\tilde f}$ is well defined. For all such $q \in U$ let $[q]_{\tilde f}$ denote the set

$$[q]_{\tilde f} = \left\{q' \in O \mid \mathrm{dist}_{\hat N}\left(\gamma_{\tilde f}(q, c_{\tilde f}(q)),\, q'\right) = c_{\tilde f}(q)\right\}.$$

For any closed subset $A \subset O$ the set $\{q \in U \mid [q]_{\tilde f} \cap A \neq \emptyset\}$ is a closed subset of U. Hence $V = \{q \in U \mid [q]_{\tilde f} \subset U\}$ is an open subset of O. Furthermore $W = \{\gamma_{\tilde f}(q,t) \mid q \in V \text{ and } 0 \le t \le c_{\tilde f}(q)\}$ is a submanifold of $\hat N$ with boundary V. By construction $\hat f_W$ is a local one-sided Alexandrov embedding. By choice of R, V is contained in the image of $B(p,R)$ under i. □

Corollary 5.12. *Let $f : N \to \mathbb{S}^3$ be a one-sided Alexandrov embedding with constant mean curvature $H \le H_{\max}$ and principal curvatures bounded by $\kappa_{\max}$. Assume that $\tilde f : M \to \mathbb{S}^3$ is an immersion of the boundary $M = \partial N$ with constant mean curvature $\tilde H \ge 0$ obeying* (5.12) *for all $q \in M$ instead of all $q \in B(p,R)$ with ϵ as in Lemma 5.11. Then $\tilde f$ extends to a one-sided Alexandrov embedding from N to $\mathbb{S}^3$.*

Proof. We apply Lemma 5.11 to all $p \in M$ and obtain a covering of M by open subsets V, which are boundaries of local one-sided Alexandrov embeddings. From (5.12) we deduced in the proof of Lemma 5.11 a bound on both principal curvatures of f and $\tilde f$. Due to Lemma 5.9 this implies a uniform bound C_2 on the chord-arc ratio. The choice of the radius R in Lemma 5.11 ensures that for any $p \in M$ the constructed local one-sided Alexandrov embedding nearby p is not affected by the immersion $\tilde f$ restricted to $M \setminus B(p,R)$. The corresponding local one-sided Alexandrov embeddings fit together to an Alexandrov embedding $\tilde f : \tilde N \to \mathbb{S}^3$ of a submanifold $\tilde N \subset \hat N$ with boundary $\partial \tilde N = M$. It remains to show that $\tilde N$ is complete with respect to the Riemannian metric induced by $\tilde f$. Due to Rosenberg's Lemma 5.8 the cut locus function is uniformly bounded by c from below. For all $r < c$, the submanifolds

$$\{\gamma_f(p,t) \in \hat N \mid p \in M \text{ and } -r \le t \le c_f(p)\} \subset \hat N$$

are complete with respect to the Riemannian metric induced by $\hat{f}$. By construction, the Riemannian manifold $\tilde{N}$ with the metric induced by $\tilde{f}$ is a closed submanifold of one of these complete submanifolds of $\hat{N}$, and therefore also complete. □

6. Deformation of CMC cylinders in $\mathbb{S}^3$

In this section we consider the subspace in the moduli space of periodic finite type solutions of the sinh-Gordon equation that contain the spectral data of CMC cylinders in $\mathbb{S}^3$, and begin with an investigation of the subset of spectral data of one-sided Alexandrov embedded CMC cylinders. We shall determine all continuous deformations of spectral data, which preserve the one-sided Alexandrov embeddedness. In order to do this, we combine the description of finite type CMC cylinders in $\mathbb{S}^3$ in terms of polynomial Killing fields with the investigation of one-sided Alexandrov embeddings. From Definition 3.1 and 4.6 we immediately conclude

Corollary 6.1. *A covering map $\kappa \mapsto \Delta(\kappa)$ satisfying conditions* (G)-(K) *corresponds to the spectral data of a* CMC *cylinder if and only if*

(L) *there are two branch points κ_0, κ_1 on the real part (i.e the fix point set of the involution ρ) at which $\Delta(\kappa_0) = \Delta(\kappa_1) = \pm 2$.* □

From the point of view of the covering map (4.2) the two distinguished points are two real branch points at $\Delta = \pm 2$, which are specified by the sheets they connect. A continuous deformation of a covering map (4.2) preserves condition (L), if the two distinguished real branch points at $\Delta = \pm 2$ are not moved and stay inside the real part. The coordinates of the two distinguished points κ_0, κ_1 are given by a parametrization of the cover $\kappa \mapsto \Delta(\kappa)$, which is unique up to Möbius transformations (3.1). As a consequence of conditions (G)-(I) the covering map is biholomorphic to $\mathbb{CP}^1$, and condition (J) equips it with an antiholomorphic involution. In general it is difficult to read off the coordinates from the parameters of the moduli space, that is the values of Δ at the branch points. An interesting parameter is the value of the mean curvature $H = (1 + \kappa_0\kappa_1)/(\kappa_0 - \kappa_1)$.

Lemma 6.2. *For continuous deformations of covering maps* (4.2) *preserving conditions* (G)-(L) *the mean curvature depends continuously on the deformation parameter.*

Proof. We describe the compactification of the covering space of Δ by an open cover $U_0, \ldots, U_n$. Here $U_1, \ldots, U_n$ are small disjoint discs around all branch point of Δ which are to be moved, and U_0 is the complement of the union of smaller closed discs $A_j \subset U_j$ such that each A_j is still a neighbourhhod of the corresponding branch point. Let w_j be a holomorphic coordinate on U_j, such that $\Delta - \Delta_j$ is a polynomial $P_j(w_j)$ with respect to w_j. The coefficients of P_j are the parameters of the continuous deformations. For sufficiently small values of these parameters the covering space is a compact Riemann surface biholomorphic to $\mathbb{CP}^1$. We describe such a biholomorphic map Φ by a meromorphic function of degree one. This is unique up to Möbius transformations.

We need to show that if we eliminate the freedom of Möbius transformations by imposing three additional conditions, that Φ depends continuously on the deformation parameter. It suffices to show that a sequence of meromorphic functions Φ_k corresponding to a convergent sequence of parameters converges to the limit Φ that corresponds to the limit of the parameters. For this it suffices to show that any such sequence of meromorphic functions contains a convergent subsequence, and that the limit is the unique limit that corresponds to the limit of the parameters. Pick on each open set U_j a Möbius transformation M_j whose composition with Φ maps the branch point of Δ contained in U_j to the origin, and with derivative at the branch point is equal to one, and such that each one of the two points $\kappa = \pm i$ is mapped to ∞. Since the

space of schlicht functions is compact (see Proposition 7.15 in Section 14.7 of Conway [14]), a subsequence of $M_j \circ \Phi_k$ converges to a biholomorphic map $U_j \to \mathbb{C}$. Since the parameters converge, the limit is equal to $M_j \circ \Phi$. ☐

The special situation $H = \infty$ corresponding to CMC cylinders in $\mathbb{R}^3$ has a simple description. In the case of CMC cylinders in $\mathbb{R}^3$, the above correspondence via spectral curves also holds with one notable exception. The closing condition for CMC cylinders in $\mathbb{R}^3$ require one value $\kappa_0 \in \mathbb{R}$ at which the monodromy is equal to $\pm\mathbb{1}$ and its derivative vanishes there [16]. Assume we have the spectral data for a CMC cylinder in $\mathbb{S}^3$ in which κ_0 and κ_1 coalesce. Then the mean curvature is infinite and the cylinder is shrunk to a point. A blow-up then yields a CMC cylinder in $\mathbb{R}^3$, see Umehara and Yamada [69].

Corollary 6.3. *A covering map satisfying conditions* (G)-(K) *corresponds to a* CMC *cylinder in* $\mathbb{R}^3$ *if and only if there is a real branch point of order at least 3.* ☐

Let us first show that all one-sided Alexandrov embedded CMC cylinders in $\mathbb{S}^3$ of finite type obey the assumptions of Rosenberg's Lemma 5.8 and the chord-arc bound in Lemma 5.9.

Lemma 6.4. *For every compact subset $\mathcal{K} \subset \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ and $H_{\max} > 0$ there exist constants $\kappa_{\max} > 0$ and $D > 0$ with the following property: If $f : \mathbb{R}^2 \to \mathbb{S}^3$ is an immersion of finite type with constant mean curvature $|H| \leq H_{\max}$, whose polynomial Killing field ζ takes at some point $p \in \mathbb{R}^2$ a value $\zeta(p) \in \mathcal{K}$. Then the absolute values of both principal curvatures of f are uniformly bounded by $\kappa_{\max}$, and the second fundamental form $\mathfrak{h}$ of f satisfies* (5.7).

Proof. The coefficients of $a(\lambda) = -\lambda \det(\xi(\lambda))$ depend continuously on $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$. Hence on every compact subset $\mathcal{K} \subset \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ these coefficients are uniformly bounded. In particular, the union $\hat{\mathcal{K}}$ of all isospectral sets $\mathcal{K}_a$ of Lemma 2.4 not disjoint from $\mathcal{K}$ is compact too. If $f : \mathbb{R}^2 \to \mathbb{S}^3$ is a CMC immersion of finite type, whose polynomial Killing field ζ takes at some $p \in \mathbb{R}^2$ a value in $\mathcal{K}$, then the other values belong to the corresponding isospectral set. Since all derivatives of f at some $p \in \mathbb{R}^2$ depend continuously on $\zeta(p)$, the lemma follows. ☐

Lemma 6.5. *For all compact subsets $\mathcal{K} \subset \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$, $\epsilon > 0$, $R > 0$ and $H_{\max} > 0$ there exists $\delta > 0$ with the following property: If $(\zeta,\, \lambda_0,\, \lambda_1)$ and $(\tilde{\zeta},\, \tilde{\lambda}_0,\, \tilde{\lambda}_1)$ are polynomial Killing fields and marked points of two* CMC *cylinders $f,\, \tilde{f} : M \to \mathbb{S}^3$ obeying at some point $p \in M$*

$$\zeta(p) \in \mathcal{K} \text{ and } |H| = \left|\frac{\lambda_0 + \lambda_1}{\lambda_0 - \lambda_1}\right| \leq H_{\max} \quad \text{or} \quad \tilde{\zeta}(p) \in \mathcal{K} \text{ and } |\tilde{H}| = \left|\frac{\tilde{\lambda}_0 + \tilde{\lambda}_1}{\tilde{\lambda}_0 - \tilde{\lambda}_1}\right| \leq H_{\max} \tag{6.1}$$

$$\|\zeta(p) - \tilde{\zeta}(p)\| < \delta\,, \qquad |\lambda_0 - \tilde{\lambda}_0| < \delta \text{ and } |\lambda_1 - \tilde{\lambda}_1| < \delta. \tag{6.2}$$

Then the corresponding immersions f and $\tilde{f}$ satisfy (5.12) *on $B(p,\, R) \subset M$.*

Proof. Since $\Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ is an open subset of $\Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})$, there exists a $\delta > 0$ and a compact subset of $\Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$, containing all balls $B(\xi, \delta)$ with $\xi \in \mathcal{K}$. Furthermore, due to Lemma 6.4 there exists also a compact subset $\hat{\mathcal{K}} \subset \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ containing all isospectral sets $\mathcal{K}_a$ not disjoint from $\cup_{\xi \in \mathcal{K}} B(\xi, \delta)$. Due to Lemma 6.4 for given $\mathcal{K} \subset \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ and $H_{\max} > 0$ there exists $\delta > 0$, $\kappa_{\max} > 0$ and $D > 0$, such that the conditions (6.1) and (6.2) imply that the corresponding immersions f and $\tilde{f}$ have principal curvatures bounded by $\kappa_{\max}$ and second fundamental forms with covariant derivatives bounded by (5.7). In particular there exists $C > 0$ such that the Riemannian metrics induced by f and $\tilde{f}$ on M are uniformly bounded by C times the Euclidean metric on the conformal parametrization $\mathbb{R}^2$. The abelian group $\mathbb{R}^2$ of spatial translations acts continuously on all elements of $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$. Therefore all elements of the closed ball $B(0, CR) \subset \mathbb{R}^2$ act uniformly continuous on $\hat{K}$. Hence for all $\tilde{\epsilon} > 0$ there exists

$\delta > 0$, such that the inequalities (6.2) imply $\|\zeta(q) - \tilde{\zeta}(q)\| < \tilde{\epsilon}$ for all $q \in B(p, R)$. Since all derivatives of $f(q)$ and $\tilde{f}(q)$ depend continuously on $\zeta(q)$ and $\tilde{\zeta}(q)$, respectively, the lemma follows. $\square$

Lemma 6.6. *For all compact subsets $\mathcal{K} \subset \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ and $H_{\max} > 0$ there exists $\delta > 0$ with the following property: Let $\tilde{f} : M \to \mathbb{S}^3$ be a* CMC *cylinder with non-negative mean curvature $\tilde{H}$ (with respect to normal uniquely determined by the orientation of M) and with polynomial Killing field $\tilde{\zeta}$ and marked points $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$. If for all $p \in M$ there exists a one-sided Alexandrov embedding $f : N \to \mathbb{S}^3$ with polynomial Killing field ζ and marked points λ_0 and λ_1 obeying conditions* (6.1) *and* (6.2)*, then $\tilde{f}$ extends to a one-sided Alexandrov embedding $\tilde{f} : \tilde{N} \to \mathbb{S}^3$.*

Proof. As in the proof of Lemma 6.5 there exists $\delta > 0$ and a compact subset $\hat{\mathcal{K}} \subset \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ containing all isospectral sets not disjoint from $\cup_{\xi \in \mathcal{K}} B(\xi, \delta)$. Due to Lemma 6.4 there exists a bound $\kappa_{\max}$ on the absolute values of the corresponding principal curvatures. From Lemma 5.11 we get a $R > 0$ and an $\epsilon > 0$ and from Lemma 6.5 a $\delta > 0$, such that the immersions f and $\tilde{f}$ obey (5.12) on $B(p, R)$ if the corresponding polynomial Killing fields ζ and $\tilde{\zeta}$ and marked points (λ_0, λ_1) and $(\tilde{\lambda}_0, \tilde{\lambda}_1)$ obey (6.1) and (6.2). Therefore these conditions imply that all $p \in M$ have a neighbourhood, on which $\tilde{f}$ extends to a local one-sided Alexandrov embedding. We remark that due to the uniform bound on the chord-arc ratio from Lemma 5.9 and the choice of $R > 0$ in Lemma 5.11 these local one-sided Alexandrov embeddings are not affected by the restriction of $\tilde{f}$ to $M \setminus B(p, R)$. Due to Corollary 5.4 these local one-sided Alexandrov embeddings can be glued together to an immersion $\tilde{f} : \tilde{N} \to \mathbb{S}^3$ with boundary $\partial\tilde{N} = M$. A subsequence of any Cauchy sequence in $\tilde{N}$ with Riemannian metric induced by $\tilde{f}$ is contained in one of the local Alexandrov embeddings from O to $\mathbb{S}^3$. Hence $\tilde{N}$ with the Riemannian metric induced by $\tilde{f}$ is complete. Since $\tilde{f}$ has non-negative mean curvature, it is a one-sided Alexandrov embedding. $\square$

Definition 6.7. *For $g \in \mathbb{N}_0$ let $\mathcal{C}_g$ denote the subset of triples $(\xi, \lambda_0, \lambda_1) \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times \times \mathbb{S}^1 \times \mathbb{S}^1$ containing the initial values (not necessarily without roots) together with both marked points of finite type* CMC *cylinders $f : M \to \mathbb{S}^3$. Let $\mathcal{C}^+_g$ denote the subset of triples $(\xi, \lambda_0, \lambda_1) \in \mathcal{C}_g$ corresponding to finite type* CMC *cylinders $f : M \to \mathbb{S}^3$ with non-negative mean curvature (with respect to the normal determined by the orientation of M).*

Theorem 6.8. *Let $\xi \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ be the initial value and (λ_0, λ_1) the marked points of a one-sided Alexandrov embedded* CMC *cylinder $f : N \to \mathbb{S}^3$. Then a* CMC *cylinder of finite type $\tilde{f} : M \to \mathbb{S}^3$ with non-negative mean curvature and with initial value $\tilde{\xi} \in \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ and marked points $(\tilde{\lambda}_0, \tilde{\lambda}_1)$ extends to one-sided Alexandrov embedding, if one of the following conditions are satisfied:*

- (M) *The polynomial $a(\lambda) = -\lambda \det(\xi)$ has $2g$ pairwise distinct roots and $\tilde{\xi}$ belongs to the same isospectral set $\mathcal{K}_a \subset \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ as ξ with the same marked points.*
- (N) *The initial value ξ has only unimodular roots and $\tilde{\xi}$ belongs to the same isospectral set $\mathcal{K}_a \subset \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ as ξ with the same marked points.*
- (O) *The polynomial $a(\lambda) = -\lambda \det(\xi)$ has $2g$ pairwise distinct roots and $(\tilde{\xi}, \tilde{\lambda}_0, \tilde{\lambda}_1)$ belongs to the same connected component of $\mathcal{C}^+_g$ as $(\xi, \lambda_0, \lambda_1)$.*
- (P) *The initial value ξ has only unimodular roots and $(\tilde{\xi}, \tilde{\lambda}_0, \tilde{\lambda}_1)$ belongs to the same connected component of $\mathcal{C}^+_g$ as $(\xi, \lambda_0, \lambda_1)$.*

In particular, real continuous deformations of spectral curves with non-negative mean curvature and decreasing G (4.9) *preserve one-sided Alexandrov embeddedness.*

Proof. The initial values ξ and $\tilde{\xi}$ extend to unique polynomial Killing fields ζ and $\tilde{\zeta}$, respectively.

We first show (M). We obtain the map $f : M \to \mathbb{S}^3$ by composing an affine immersion of $M \cong \mathbb{S}^1 \times \mathbb{R}$ into the real part of the Jacobian with the Sym-Bobenko formula. In case that a has pairwise distinct roots, the real part of the Jacobian is a real g dimensional compact torus, and acts freely and transitively on the corresponding isospectral set $\mathcal{K}_a$. Due to Lemma 2.4 the isospectral sets are compact. A continuous action of a finite dimensional Lie group on a compact space is uniformly continuous. Hence for all $\tilde{f}$ in a small neighbourhood of f in the isospectral set (5.12) is satisfied on $q \in M$ instead of $q \in B(p, R)$. Due to Lemma 6.4 and Rosenberg's Lemma 5.8 the cut locus function c_f is uniformly bounded from below. Corollary 5.12 implies that the set of one-sided Alexandrov embedded CMC cylinders in the isospectral set is open and closed. This proves (M).

Now we prove (N). If the polynomial a of the initial value ξ has roots of higher order, then the corresponding isospectral set $\mathcal{K}_a \subset \Lambda^g_{-1}\mathfrak{sl}_2(\mathbb{C})^\times$ has a stratification into strata, on which the real part of the Jacobian acts transitively. Due to Lemma 2.6 the initial value ξ belongs to the stratum of highest dimension, which is dense in the whole isospectral set. In this case the arguments above concerning the case (M) carry over and show that whenever a stratum contains a one-sided Alexandrov embedded CMC cylinder, then all elements of this stratum correspond to one-sided Alexandrov embedded CMC cylinders. Lemma 6.6 now implies (N).

Now we prove part (O). Again we show that the set of all spectral curves of arithmetic genus g (i.e. the set of polynomials $a(\lambda) = -\lambda \det(\xi)$ of degree $2g$), whose isospectral sets contain one-sided Alexandrov embedded CMC cylinders, is open and closed. Since the isospectral sets are compact, the Hausdorff distance between the isospectral sets defines a metric on the space of spectral data of given arithmetic genus. Lemma 6.6 implies that the isospectral sets of one-sided Alexandrov embeddings are closed and open in $\mathcal{C}_g^+$. This proves (O).

Due to (N) the arguments of the proof of (O) carry over to (P), and concludes the proof. □

7. One-sided Alexandrov embedded CMC cylinders in $\mathbb{S}^3$ of finite type

In this concluding chapter we classify one-sided Alexandrov embedded CMC cylinders of finite type in the 3-sphere. We first show in Theorem 7.1 that there is a 1-parameter family of flat one-sided Alexandrov embedded CMC cylinders, and a 2-parameter family of one-sided Alexandrov embedded rotational CMC cylinders of spectral genus $g = 1$. We then prove (Lemma 7.3) that an arbitrary one-sided Alexandrov embedded CMC cylinder of finite type in the 3-sphere can be continuously deformed into a flat one-sided Alexandrov embedded CMC cylinder, while preserving the condition of one-sided Alexandrov embeddedness throughout the whole deformation. We next prove (Lemma 7.4) that a surface with bubbletons is not one-sided Alexandrov embedded. Putting the above results together in Theorem 7.6 gives us the following classification: A one-sided Alexandrov embedded CMC cylinder of finite type in the 3-sphere is a surface of revolution.

Since an embedded CMC torus in the 3-sphere is covered by a one-sided Alexandrov embedded CMC cylinder of finite type, this result confirms the conjecture by Pinkall and Sterling [51] that the only embedded CMC tori in the 3-sphere are tori of revolution. In particular, since by a result by Hsiang and Lawson [27], the only embedded minimal torus of revolution is the Clifford torus, this affirms the Lawson Conjecture. We conclude the paper with a generalization of an 'unknottedness' result by Lawson [44], and show that for a one-sided Alexandrov embedded CMC cylinder of finite type in the 3-sphere, the 3-manifold is diffeomorphic to the cartesian product $\overline{\mathbb{D}} \times \mathbb{R}$, where $\overline{\mathbb{D}}$ is the closed unit disk.

We first turn our attention to the spectral data of one-sided Alexandrov embedded rotational CMC cylinders of spectral genus $g \leq 1$.

Theorem 7.1. *There exists a family of spectral data of one-sided Alexandrov embedded* CMC *cylinders with spectral genus $g \leq 1$ parameterized by the mean curvature $H \geq 0$ and $\alpha \in [0, 1)$. The corresponding spectral curves are given by $a(\kappa) = \kappa^2 + \alpha$. The boundary of the moduli $(H,\, \alpha) \in [0,\, \infty) \times [0,\, 1)$ consists of*

flat cylinders in $\mathbb{S}^3$: $H \in [0,\, \infty]$, $\alpha = 0$,
minimal cylinders in $\mathbb{S}^3$: $H = 0$, $\alpha \in [0,\, 1)$,

In case of flat cylinders in $\mathbb{S}^3$ the spectral genus zero curves have a real double point at $\kappa = 0$ and no other real double points. This case contains all spectral data of one-sided Alexandrov embedded CMC *cylinders of geometric genus zero. In case $\alpha \neq 0$ the spectral curves have no real double points. As $\alpha \to 0$ the two branch points coalesce into a double point at $\kappa = 0$.*

Proof. We recall the spectral data for cylinders of spectral genus zero, and the expressions for $\ln\mu$ of (4.4) and $d\ln\mu$ of (4.5). At $\kappa = \pm\kappa_0$ we require the closing conditions that $\ln\mu \in \pi i\mathbb{Z}$. Thus for some integers $m,\, n \in \mathbb{Z}$ we have

$$\begin{aligned} \ln\mu\big|_{\kappa_0} \in \pi i\mathbb{Z} &\Longleftrightarrow 4(b_0\kappa_0 - b_1)^2 = n^2(\kappa_0^2 + 1)\,, \\ \ln\mu\big|_{-\kappa_0} \in \pi i\mathbb{Z} &\Longleftrightarrow 4(b_0\kappa_0 + b_1)^2 = m^2(\kappa_0^2 + 1)\,. \end{aligned} \tag{7.1}$$

We make the following claim: If a cylinder is one-sided Alexandrov embedded then $m = n = \pm 1$ (for this ensures that the surface is simply wrapped with respect to the rotational period). To prove this claim first note that any spectral genus zero cylinder is a covering of a flat embedded torus. The complement of this flat embedded torus with respect to $\mathbb{S}^3$ consists of two connected components $\mathcal{D}_\pm$, both diffeomorphic to $\mathbb{D} \times \mathbb{S}^1$. Assume the mean curvature vector points into $\mathcal{D}_+$. For a one-sided Alexandrov embedded CMC cylinder the extension $f : N \to \mathbb{S}^3$ is a surjective immersion onto the closure $\bar{\mathcal{D}}_+$ of $\mathcal{D}_+$. Hence this map is a covering map. The fundamental group of $\bar{\mathbb{D}} \times \mathbb{S}^1$ is isomorphic to $\mathbb{Z}$. Now the covers of a topological space are in one-to-one correspondence with subgroups of the fundamental group [63, §14], and all non-trivial subgroups of $\pi_1(\bar{\mathbb{D}} \times \mathbb{S}^1) \cong \mathbb{Z}$ correspond to compact covers. Hence the only non-compact cover of $\bar{\mathbb{D}} \times \mathbb{S}^1$ is the universal cover $\bar{\mathbb{D}} \times \mathbb{R}$. Therefore f is the universal covering map. In particular the period of the cylinder is a primitive period in the kernel of

$$H_1(\mathbb{S}^1 \times \mathbb{S}^1,\, \mathbb{Z}) \to H_1(\bar{\mathbb{D}} \times \mathbb{S}^1,\, \mathbb{Z})\,.$$

This implies $m = n = \pm 1$ and proves the claim.

Returning to the proof of the theorem, then (7.1) with $m^2 = n^2 = 1$ simplify to $b_0 b_1 \kappa_0 = 0$. Thus if $\kappa_0 \neq 0$ then $b_0 b_1 = 0$. Now $|H|$ attains all values in $[0,\, \infty)$ when either κ_0 ranges over values $\kappa_0 \in (0,\, 1]$, or when κ_0 ranges over values $\kappa_0 \in [1,\, \infty)$. Hence we can pick a κ_0 either from $(0,\, 1]$ or from $[1,\, \infty)$. The Möbius transformation (3.1) $\kappa \mapsto -\frac{1}{\kappa}$ interchanges these two intervals and the two cases $b_0 = 0$ and $b_1 = 0$. Hence we may assume without loss of generality that $b_1 = 0$. Then $4\kappa_0^2 b_0^2 = \kappa_0^2 + 1$.

Now we claim that with this choice only $\kappa_0 \in [1,\, \infty)$ correspond to one-sided Alexandrov embedded cylinders. In fact, we have seen above that the period of the cylinder should correspond to an element in the kernel of $H_1(\partial\mathcal{D}_+, \mathbb{Z}) \to H_1(\bar{\mathcal{D}}_+, \mathbb{Z})$. The Möbius transformation (3.1) $\kappa \mapsto -\frac{1}{\kappa}$ interchanges the cases $\kappa_0 \in (0,\, 1]$ and $\kappa_0 \in [1,\, \infty)$. Hence we may consider both families of flat cylinders as two copies of one family of tori considered as cylinders with respect to different periods. In the limit $\kappa_0 \to 0$ the length of the period tends to infinity, and in the limit $\kappa_0 \to \infty$ the length of the period is bounded. Hence the period of the first family are the rotation period of $\bar{\mathcal{D}}_-$, i.e. a primitive element of the kernel of $H_1(\partial\mathcal{D}_-, \mathbb{Z}) \to H_1(\bar{\mathcal{D}}_-, \mathbb{Z})$ and therefore the translation period of $\bar{\mathcal{D}}_+$. The period of the second family are the rotation period of $\bar{\mathcal{D}}_+$. Therefore only the family $\kappa_0 \in [1, \infty)$ corresponds to one-sided Alexandrov embedded cylinders.

For a double point κ this means that $(\kappa_0^2+1)\,\kappa^2 = n^2(\kappa^2+1)\,\kappa_0^2$ for some $n\in\mathbb{Z}$. We have

$$0 \le \frac{\kappa^2}{\kappa^2+1} \le 1 \quad \text{for real } \kappa \text{ and} \qquad\qquad \frac{1}{2} \le \frac{\kappa_0^2}{1+\kappa_0^2} < 1 \quad \text{for } \kappa_0 \in [1,\,\infty).$$

Hence for $\kappa_0 \in [1,\,\infty)$ this family has only the real double points $\kappa = \pm\kappa_0$ with $n = \pm 1$ and $\kappa = 0$ with $n = 0$. Due to the closing condition (ii) in Definition 3.1 the former has to be preserved. If we open the latter double points then $\ln\mu$ remains a meromorphic function on the genus 1 spectral curve. We thus altogether obtain families parameterized by $\kappa_0 \in [1,\,\infty)$ and $\alpha \in [0,1)$:

$$\begin{aligned} a(\kappa) &= \kappa^2+\alpha\,, & \nu^2 &= (\kappa^2+1)(\kappa^2+\alpha)\,,\\ \ln\mu &= 2\pi i\, b_2\,\frac{\kappa^2+\alpha}{\nu}\,, & d\ln\mu &= 2\pi i\, b_2\,\frac{\kappa^2+1-\alpha}{(\kappa^2+1)\nu}\,. \end{aligned}$$

At $\pm\kappa_0$ the closing conditions $\ln\mu = \pm\pi i$ must hold and thus $4(\kappa_0^2+\alpha)^2 b_2^2 = (\kappa_0^2+\alpha)(\kappa_0^2+1)$ giving $4(\kappa_0^2+\alpha)\,b_2^2 = \kappa_0^2+1$. Therefore double points have to satisfy

$$\frac{\kappa^2+\alpha}{\kappa^2+1} = n^2\frac{\kappa_0^2+\alpha}{\kappa_0^2+1} \quad \text{for some } n\in\mathbb{Z}\,.$$

But for $\kappa_0 \in [1,\,\infty)$ and $\alpha \in (0,1)$ this equation has no real solutions besides $n = \pm 1$ and $\kappa = \pm\kappa_0$. Hence these families have no real double points. $\square$

In addition to the two boundary components there exists two limiting cases: When $H = \infty$, $\alpha \in [0,\,1]$ we obtain unduloidal CMC cylinders in $\mathbb{R}^3$; When $H \in [0,\infty)$, $\alpha = 1$, the resulting surfaces are *chain of spheres*. Note that as a consequence of Theorem 7.1, when $\alpha \neq 0$ there is no real branch point at $\Delta = \pm 2$. Hence it is only possible to increase the genus by opening two conjugate double points in this case.

Lemma 7.2. *Every spectral data of a one-sided Alexandrov embedded* CMC *cylinder in* $\mathbb{S}^3$ *can be deformed into spectral data of a one-sided Alexandrov embedded* CMC *cylinder in* $\mathbb{S}^3$ *with arbitrary large mean curvature* H *by a continuous deformation that increases* H *and decreases* G (4.9)*.*

Proof. The real part is the fixed point set of the anti-linear involution ρ. It is a compact one-dimensional submanifold of the domain of Δ, which is isomorphic to $\mathbb{S}^1$. Since the values of Δ at both κ_0, κ_1 are equal to ± 2 both segments of the real part between κ_0 and κ_1 contain an odd number of branch points of Δ (counted with multiplicities). The simultaneous movements of κ_0 and κ_1 directed inward to one of these two segments, increases the mean curvature with respect to the inward normal (determined by the orientation of M). This segment we call the **short segment**, and the other segment the **long segment**. We use a Möbius transformation to ensure that $\kappa = \infty$ is contained in the long segment.

We claim, that the movement of a simple real branch point of Δ within the short segment, which increases the value of Δ at the simple branch point, if it is a local minimum, and decreases the value of Δ otherwise, increases the mean curvature. In fact, due to (3.7) the signs of $\dot\kappa_i$ are equal to the signs of $-c(\kappa_i)/b(\kappa_i)$. We may assume that the simple branch point is a zero of b. Hence c has the same zeroes as b with the exception of the simple branch point, whose value of Δ is changed. Hence the sign of the function $\kappa \mapsto -c(\kappa)/b(\kappa)$ on the short segment changes the sign only at the simple branch point. The two points κ_0 and κ_1 sit on different sides of this simple branch point. Hence the sign of $\dot H$ does not depend on the position of κ_0 and κ_1, as long they stay on the corresponding side of the simple branch point. If κ_0 and κ_1 sit nearby the simple branch point, then due to (4.3) and (3.7) the movement, which increases the value

of Δ at local minima and decreases the values of Δ at local maxima, moves κ_0 and κ_1 towards each other. This proves the claim.

Due to condition (F) the real branch point at $\Delta = \pm 2$ have odd orders. As a first step we shall move each such real branch point of odd order in the short segment at $\Delta = \pm 2$ by a small movement into $\Delta \in (-2, 2)$. If we choose a to have at all real branch points at $\Delta = \pm 2$ roots of the same order as $\Delta^2 - 4$, then b has at all real branch points roots of the same order as Δ'. Moreover, the corresponding $\frac{c}{b}$ changes the sign at the real branch point, and the arguments concerning simple real branch points in the short segment carry over. This deformation shortens the short segment, and therefore increases the mean curvature. These movements do not change G (4.9) and increases the geometric genus into G. All other deformations shall decrease the geometric genus.

In a second step we increase the values of Δ of all simple real branch points within the short segment, which are local minima, and decrease the values of Δ of all simple real branch points within the short segment, which are local maxima. This again shortens the short segment and increases the mean curvature. All higher order real branch points within the short segment we deform into pairs of complex conjugate branch points close to the real part and eventually one simple real branch point. Similar as in the third step of the proof of Lemma 4.5 we can achieve with increasing mean curvature a situation with only one simple real branch point within the short segment. If we move this real branch point arbitrarily close to the two marked points κ_0 and κ_1, then the mean curvature becomes arbitrarily large with decreasing G (4.9). □

The arguments show that we can deform the spectral data of a one-sided Alexandrov embedded CMC cylinder in $\mathbb{S}^3$ into the spectral data of a CMC cylinder in $\mathbb{R}^3$. If the two marked points κ_0 and κ_1 coalesce, then the corresponding surface in $\mathbb{S}^3$ shrinks to a point. But if we enlarge simultaneously all distances of $\mathbb{S}^3$ on an appropriate scale, then the corresponding CMC cylinders converge to a CMC cylinder in $\mathbb{R}^3$. In fact, the enlargement of the distances of $\mathbb{S}^3$ corresponds to spheres in $\mathbb{R}^4$ of enlarged radius and therefore also to three-dimensional space forms of diminishing constant sectional curvature. Our arguments can be used to show, that all spectral data of one-sided Alexandrov embedded CMC cylinders in $\mathbb{S}^3$ can be continuously deformed within this class into spectral data of one-sided Alexandrov embedded CMC cylinders in $\mathbb{R}^3$. In [39] it is shown that they have genus at most equal to one. Hence we could have used [39] instead of the subsequent Lemmas to prove the main Theorem 7.6. Conversely, our arguments can be used to show that all one-sided Alexandrov embedded CMC cylinders in $\mathbb{R}^3$ of finite type are Delaunay surfaces. We expect that our arguments extend to all one-sided Alexandrov embedded CMC cylinders in $\mathbb{R}^3$ with constant Hopf differential (compare [45, 58, 67]).

Lemma 7.3. *All spectral data of one-sided Alexandrov embedded* CMC *cylinders in* $\mathbb{S}^3$ *can be continuously deformed within the class of one sided Alexandrov embedded* CMC *cylinders into spectral data of flat cylinders in* $\mathbb{S}^3$ *described in Theorem 7.1.*

Proof. In a first step, due to Lemma 7.2, we can deform the spectral data of one-sided Alexandrov embedded CMC cylinders in $\mathbb{S}^3$ continuously with decreasing G (4.9) and increasing H into spectral data with only one simple real branch point in the short segment and arbitrary large mean curvature.

In a second step we apply a similar procedure on the real branch points in the long segment. We can control the mean curvature by moving the only real branch point within the short segment closer to the two marked points κ_0 and κ_1. Hence we obtain spectral data with only two real branch points arbitrary close to the two marked points κ_0 and κ_1 with arbitrary large mean curvature.

In a third step we apply the deformation described in the fourth step of the proof of Lemma 4.5. We obtain spectral data of genus at most equal to one with two simple real branch points very close to the two branch points at κ_0 and κ_1.

In a fourth step we move the real branch point, which connects the sheets with lower labels, away from the two real branch points at κ_0 and κ_1 along $\Delta \in [-2, 2]$ to and fro until it reaches the place $\Delta = 2$ on the sheets with labels $1^{\pm}$. We claim that we can preserve positive mean curvature by moving, if necessary, the other real branch point in the other direction along $\Delta \in [-2, 2]$ to and fro. The function Δ is equal to $\mu + \mu^{-1} = 2\cosh(\ln\mu)$. Hence it suffices to show that we preserve the values of κ at both marked points by changing the values of $\ln\mu$ at the other real branch point. The polynomial c, which corresponds to the deformation preserving the mean curvature is proportional to $c(\kappa) = (\kappa - \kappa_0)(\kappa - \kappa_1)$. The derivatives $\dot{\ln\mu} = \dot{\mu}\mu^{-1}$ at the real branch points, that is the zeroes of the corresponding polynomial b, are equal to the values of $\frac{c(\kappa)}{\nu}$ at the real branch points due to (3.5). We remark that since $d\ln\mu$ vanishes at these real branch points any change of the values of κ at these real branch points has no influence on the derivatives $\dot{\ln\mu}$ of the values of $\ln\mu$ at these real branch points. As long as the real branch point has not reached $\Delta = 2$ on the sheets with labels $1^{\pm}$, the function ν has no roots on the real part. The two integrals of $d\ln\mu$ along the short and long segments are preserved under the deformation. Both real roots of b are local extrema of $\frac{1}{2\pi i}\ln\mu$ on the real part. During the deformation the value of $\frac{1}{2\pi i}\ln\mu$ is increased in case of a local maximum, and decreased in case of a local minimum. Hence during the deformation one of the real roots of b stays in the short segment, while the other real root stays in the long segment. Therefore we can move, without changing the mean curvature, the real branch point arbitrarily close to $\Delta = 2$ on the sheets with labels $1^{\pm}$. Due to Lemma 6.2 there exist for any $\epsilon > 0$ a $\delta > 0$ such that the movement from $\Delta = 2 - \delta$ to $\Delta = 2$ changes the mean curvature by at most ϵ. This proves the claim.

Due to Theorem 6.8 the whole deformation preserves the one-sided Alexandrov embeddedness. Hence the final spectral data belong to the flat cylinders in $\mathbb{S}^3$ described in Theorem 7.1. $\square$

Lemma 7.4. *The* CMC *cylinders with bubbletons of finite type, whose* Δ (4.2) *corresponds to one of the flat cylinders described in Theorem 7.1 are not one-sided Alexandrov embedded.*

Proof. Due to Theorem 6.8 (O) it is enough to show that the CMC cylinders, which are obtained from the continuous deformation of a Δ corresponding to a flat cylinder described in Theorem 7.1 by moving two simple complex conjugate branch points from $\Delta = \pm 2$ into $\Delta \in \mathbb{C} \setminus [-2, 2]$ and increasing the genus by two, are not one-sided Alexandrov embedded. Afterwards these two places at $\Delta = \pm 2$ have no branch points. In the following first two steps we can move the unique real branch point in the short segment arbitrarily close to the two marked points and make the mean curvature arbitrarily large.

In a first step we move these branch points to the long segment of the real part between the two marked points κ_0 and κ_1, which contains the real branch point at $\Delta = 2$. Afterwards this segment contains two additional real branch points.

In a second step we move these real branch points on the real part away from each other. Then between them two new sheets become connected through a new segment of the real part. Hence we can move one of these two real branch points several times along $\Delta \in [-2, 2]$ to and fro until it reaches one of the two places at $\Delta = \pm 2$ without branch points. Hence the genus is reduced by one.

In a third step we move the only real branch point in the short segment away from the two marked points along $\Delta \in [-2, 2]$ to and fro until it reaches the other place $\Delta = \pm 2$ without other branch points. In order to preserve positivity of the mean curvature, we move the other

real root of b in the long segment away from the two marked points along $\Delta \in [-2, 2]$ to and fro. The arguments of the fourth step in the proof of Lemma 7.3 ensure this is always possible. The whole deformation has twice deformed one zero of b, and two zeroes of a into a real double point, and thus arrive at a flat cylinder of spectral genus zero. It cannot belong to the family of flat cylinders described in Theorem 7.1, since the corresponding family of spectral genus one curves described in this Theorem do not have real double points. Due to Theorem 6.8 (O)-(P) these deformations preserve one-sided Alexandrov embeddedness. Hence all spectral data obtained by adding to the flat cylinders in $\mathbb{S}^3$ described in Theorem 7.1 a bubbleton do not correspond to one-sided Alexandrov embedded CMC cylinders in $\mathbb{S}^3$. □

Example 7.5. *As an example of the technique used in the proof of Lemma 7.4 we demonstrate how to deform the simplest case of bubbletons at the branch point over $\Delta = 2$ connecting the sheets with labels $2^\pm$ and $3^\pm$: We describe the movement of the branch point connecting the sheets which have labels with superscript $+$. The other branch point is moved along the conjugate path. First we move the branch point connecting the sheets with labels 2^+ and 3^+ counter clockwise around the interval $\Delta \in [-2, 2]$ until it meets the origin $\Delta = 0$ from the lower half-plane. Our branch point now meets the conjugate branch point at the real part and connects sheets with labels 1^+ and 4^+. Further, the branch point over $\Delta = -2$ which initially connected the sheets with labels 1^+ and 2^+, now connects the sheets with labels 4^+ and 2^+, and the branch point over $\Delta = -2$ which initially connected the sheets with labels 3^+ and 4^+, now connects the sheets with labels 3^+ and 1^+. Since the two coalescing branch points are now real, we can separate them along the real part. We move one of them connecting sheets with labels 1^+ and 1^- to the left until $\Delta = -2$, and then move it further connecting the sheets with labels 3^+ and 3^- to the right until $\Delta = 2$. Now the genus is reduced to one. Next we simultaneously move the vertical cut between the sheets with labels 1^+ and 4^+ and the vertical cut between the sheets with labels 1^- and 4^- along the same way until it reaches $\Delta = -2$ and continue on until it reaches the unique real branch point in the short segment. This has the effect that over $\Delta = -2$, the branch point which previously connected the sheets with labels $4^\pm$ and $2^\pm$, now connects the sheets with labels $1^\pm$ and $2^\pm$, and the branch point over $\Delta = -2$ which previously connected the sheets with labels $3^\pm$ and $1^\pm$, now connects the sheets with labels $3^\pm$ and $4^\pm$. The vertical cuts now connect the sheets with labels $3^\pm$ and $2^\pm$. Now we move the real branch point in the short segment to $\Delta = 2$. To preserve positivity of the mean curvature, we move the other real branch point, which connects the sheets with labels 4^+ and 4^-, towards $\Delta = 2$ and possibly further to and fro. The genus is thus reduced to zero. We arrive at a constellation described in the proof of Theorem 7.1 with $\kappa_0 \in (0, 1)$, which we know not to be one-sided Alexandrov embedded.*

Theorem 7.6. *A one-sided Alexandrov embedded* CMC *cylinder of finite type in the 3-sphere is a surface of revolution.*

Proof. We will show that all spectral data of one-sided Alexandrov embedded CMC cylinders of finite type in $\mathbb{S}^3$ are described in Theorem 7.1. Due to Lemma 7.3 all spectral data corresponding to a one–sided Alexandrov embedded CMC cylinder can be continuously deformed within the class of Δ corresponding to one-sided Alexandrov embedded CMC cylinders by a combination of continuous deformations described in Theorem 6.8 (N)-(P) into one of the spectral data of the flat cylinders in $\mathbb{S}^3$ described in Theorem 7.1. Due to Lemma 7.4 the Δ corresponding to these flat cylinders in $\mathbb{S}^3$ can be continuously deformed within the class of one-sided Alexandrov embedded CMC cylinders only into the Δ described in Theorem 7.1. All CMC cylinders with bubbletons of finite type, whose Δ (4.2) are described in Theorem 7.1 can be deformed within the class of deformations preserving the one-sided Alexandrov embeddedness into the CMC cylinders with bubbletons described in Lemma 7.4. Hence they are not one-sided Alexandrov embedded. Consequently, all spectral data corresponding to one-sided Alexandrov embedded CMC cylinders in $\mathbb{S}^3$ are described in Theorem 7.1. They are all surfaces of revolution! □

Corollary 7.7. *All one-sided Alexandrov embedded* CMC *tori in the 3-sphere are tori of revolution. In particular, all embedded* CMC *tori in the 3-sphere are tori of revolution.*

Proof. Let $f : N \to \mathbb{S}^3$ be a one-sided Alexandrov embedding of a torus $M = \partial N \simeq \mathbb{T}^2$. We will show that there exists a cover $\tilde{N} \to N$, whose composition with f is a CMC cylinder which is also one-sided Alexandrov embedded. If the class in $\pi_1(M)$ of a smooth embedding $\gamma : \mathbb{S}^1 \hookrightarrow M$ belongs to the kernel of $\pi_1(M) \to \pi_1(N)$, then, due to Dehn's Lemma, γ is the boundary of a smooth embedding from the closed two-dimensional disc $\bar{\mathbb{D}} \hookrightarrow N$. The intersection number of another path $\mathbb{S}^1 \to M$ with γ in M is equal to the intersection number with the disc in N. Hence the kernel of $\pi_1(M) \to \pi_1(N)$ is isotropic with respect to the intersection form and therefore cyclic. In particular, there exists a subgroup of $\Gamma \subset \pi_1(N)$, whose pre-image under $\pi_1(M) \to \pi_1(N)$ is isomorphic to $\mathbb{Z}$. Furthermore, we can choose the subgroup Γ such that the restriction of $\pi_1(M) \to \pi_1(N)$ to the subgroup $\mathbb{Z} \subset \pi_1(M)$ is surjective onto Γ. This subgroup $\Gamma \subset \pi_1(N)$ corresponds to a covering $\tilde{N} \to N$, whose boundary $\tilde{M} = \partial \tilde{N}$ is a cylinder [63, §14]. Hence there exists a one-sided Alexandrov embedded cylinder $\tilde{f} : \tilde{N} \to \mathbb{S}^3$, which is the composition of f with a covering map. Obviously the spectral data of f and $\tilde{f}$ coincide. Due to Pinkall and Sterling [51] this cylinder is of finite type, and by Theorem 7.6, it is a surface of revolution. □

Hsiang and Lawson [27] prove that there are no embedded minimal tori of cohomogeneity one. Hence the Clifford torus is the only embedded minimal torus of revolution. By Corollary 7.7 the only embedded CMC tori are tori of revolution, and we thus affirm Lawson's conjecture.

Corollary 7.8. *The Clifford torus is the only embedded minimal torus in the 3-sphere.*

Since all one-sided Alexandrov embedded CMC cylinders in $\mathbb{S}^3$ are surfaces of revolution around a closed geodesic, the ambient 3-manifold is diffeomorphic to $\overline{\mathbb{D}} \times \mathbb{R}$, where $\overline{\mathbb{D}}$ denotes the closed unit disk. We thus have the following generalization of Lawson's 'unknottedness' result [44].

Corollary 7.9. *For all one-sided Alexandrov embedded* CMC *cylinders of finite type in the 3-sphere, the 3-manifold is diffeomorphic to the cartesian product $\overline{\mathbb{D}} \times \mathbb{R}$.*

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